Generalized and Unified Equivalences between Hardness and Pseudoentropy

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Abstract

Pseudoentropy characterizations provide a quantitatively precise demonstration of the close relationship between computational *hardness* and computational *randomness*. We prove a unified pseudoentropy characterization that generalizes and strengthens previous results for both uniform and non-uniform models of computation. Our characterization holds for a general family of entropy notions that encompasses the common notions of Shannon entropy and min entropy as special cases. Moreover, we show that the characterizations for different entropy notions can be simultaneously achieved by a single, universal function that simultaneously witnesses computational hardness and computational randomness. A key technical insight of our work is that the notion of *weight-restricted calibration* from the recent literature on algorithm fairness, along with standard computational indistinguishability (known as *multiaccuracy* in the fairness literature), suffices for proving pseudoentropy characterizations for general entropy notions. This demonstrates the power of weight-restricted calibration to enhance the classic Complexity-Theoretic Regularity Lemma (Trevisan, Tulsiani, and Vadhan, 2009) and Leakage Simulation Lemma (Jetchev and Pietrzak, 2014) and allows us to achieve an exponential improvement in the complexity dependency on the alphabet size compared to the pseudoentropy characterizations by Casacuberta, Dwork, and Vadhan (2024) based on the much stronger notion of *multicalibration*. We show that the exponential dependency on the alphabet size is inevitable for multicalibration as well as for the weaker notion of calibrated multiaccuracy.

1 Introduction

1.1 Background on Hardness–Randomness Equivalences

The close relationship between computational *hardness* and computational *randomness* is central in cryptography and complexity theory. A classic example of this relationship is Yao's equivalence between pseudorandomness and (maximal) unpredictability [Yao82], one form of which is the following:

Theorem 1.1. *Let (X, Y) be a random variable distributed on $\{0,1\}^n \times \{0,1\}^\ell$ with $\ell = O(\log n)$. Then the following are equivalent:*

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1. $(X, Y) \approx^c (X, U_\ell)$, *where* $\approx^c$ *denotes computational indistinguishability (against nonuniform polynomial-time algorithms).*

2. *For every nonuniform polynomial-time algorithm* A*,*

$$\Pr[A(X) = Y] \leq \frac{1}{2^\ell} + \mathrm{negl}(n).$$

We refer to Condition 2 as *maximal* unpredictability, because it is trivial for an efficient A to achieve prediction probability $1/2^\ell$, by just outputting a uniformly random ℓ-bit string. It is natural to ask what happens if we weaken the hardness of prediction to allow A to succeed with other probability between $1/2^\ell$ and 1. Vadhan and Zheng [VZ12, Zhe14] showed that such weak unpredictability is equivalent to Y having high *pseudo-average-min-entropy* [HLR07] given X.

Theorem 1.2. *Let* (X, Y) *be a random variable distributed on* $\{0,1\}^n \times \{0,1\}^\ell$ *with* $\ell = O(\log n)$*, and let* $k \in [0, \ell]$*. Then the following are equivalent:*

1. *There is a random variable* Z *on* $\{0,1\}^\ell$*, jointly distributed with* X*, such that* $(X, Y) \approx^c (X, Z)$ *and* $\mathsf{H}_{\mathsf{avg\text{-}min}}(Z|X) \geq k$*, where* $\mathsf{H}_{\mathsf{avg\text{-}min}}$ *denotes* average min-entropy *[DORS08]:*

$$\mathsf{H}_{\mathsf{avg\text{-}min}}(Z|X) := -\log_2 \mathbb{E}_X \left[\max_{z_X \in \{0,1\}^\ell} \Pr[Z = z_X|X] \right].$$

2. *For every nonuniform polynomial-time algorithm* A*,*

$$\Pr[A(X) = Y] \leq \frac{1}{2^k} + \mathrm{negl}(n),$$

where $\mathrm{negl}(n)$ *denotes a negligible function.*

As noted in [VZ12, Zhe14], the case when Y consists of $\ell = 1$ bits is equivalent to Holenstein's optimal version of Impagliazzo's Hardcore Lemma [Imp95, Hol05]; the novelty in their result was proving it for all $\ell = O(\log n)$.

Now it is also natural to ask what happens if we replace the average min-entropy above with other information-theoretic measures of randomness, like conditional Shannon entropy. In this case, Vadhan and Zheng [VZ12] gave the following characterization.

Theorem 1.3. *Let* (X, Y) *be a random variable distributed on* $\{0,1\}^n \times \{0,1\}^\ell$ *with* $\ell = O(\log n)$*, and let* $k \geq 0$*. Then the following are equivalent:*

1. *There is a random variable* Z *on* $\{0,1\}^\ell$*, jointly distributed with* X*, such that* $(X, Y) \approx^c (X, Z)$ *and* $\mathsf{H}_{\mathsf{Sh}}(Z|X) \geq \mathsf{H}_{\mathsf{Sh}}(Y|X) + k - \mathrm{negl}(n)$*, where* H_{Sh} *denotes the conditional Shannon entropy.*

2. *For every nonuniform, randomized polynomial-time algorithm* A*,*

$$\mathsf{D}_{\mathsf{KL}}((X, Y) \| (X, A(X))) \geq k - \mathrm{negl}(n),$$

where D_{KL} *denotes KL divergence (a.k.a. relative entropy).*

Here the hardness of $A(X)$ trying to predict the value of Y is replaced by $A(X)$ trying to get close (in KL divergence) to the conditional distribution of Y given X. Vadhan and Zheng [VZ12] used Theorem 1.3 to give a simpler construction of pseudorandom generators from one-way functions. Their pseudorandom generator construction avoids the use of hardcore bits like Goldreich–Levin [GL89], which was the traditional way of converting the hardness of inverting a one-way function into pseudoentropy, as was used in the original construction of Håstad, Impagliazzo, Levin, and Luby [HILL99].

Note that Theorem 1.3 is concerned with the *pseudoentropy gap* $k = \mathsf{H}_{\mathsf{Sh}}(Z|X) - \mathsf{H}_{\mathsf{Sh}}(Y|X)$, which measures how much *more* randomness Y appears to have to feasible observers in comparison to computationally unbounded ones. Correspondingly, even if $\mathsf{H}_{\mathsf{Sh}}(Y|X)$ is large, Y may still be considered "easy" in that an efficient $A(X)$ can sample accurately from the conditional distribution $Y|_X$. In contrast, the parameter k in Theorem 1.2 does not distinguish between the computational hardness and information-theoretic hardness in Y given X.

Aside from this difference, the two theorems are very similar in spirit and one may wonder whether there is a unified proof for them, one which ideally also generalizes to other measures of randomness (beyond min-entropy and Shannon entropy). Zheng [Zhe14] made progress in this direction, considering a wide family of entropy measures H_φ and their corresponding *Bregman divergences* D_φ (whose definition we defer to later in the introduction).

Theorem 1.4. *Suppose that $\ell = O(\log n)$ and let $\mathsf{H}_\varphi(B|A)$ be a function that takes a random variable (A, B) distributed on $\{0,1\}^n \times \{0,1\}^\ell$ and outputs the negative average over $a \leftarrow A$ of a differentiable and strictly convex function φ applied to the probability mass function of $B|A = a$. Let D_φ be the Bregman divergence associated with φ. Suppose that given a linear function Λ, the probability mass function p on $\{0,1\}^\ell$ that minimizes $\Lambda(p) + \varphi(p)$ can be computed by a nonuniform algorithm running in time $\mathsf{poly}(n)$, and that, given p, the gradient $\nabla\varphi(p)$ can also be computed accurately in time $\mathsf{poly}(n)$. Also assume that $\sup_p \|\nabla\varphi(p)\|_\infty = \mathsf{poly}(n)$.*

Let (X, Y) be a random variable distributed on $\{0,1\}^n \times \{0,1\}^\ell$ with $\ell = O(\log n)$, and let $k \geq 0$. Then the following are equivalent:

1. *There is a random variable Z on $\{0,1\}^\ell$, jointly distributed with X, such that $(X, Y) \approx^c (X, Z)$ and $\mathsf{H}_\varphi(Z|X) \geq \mathsf{H}_\varphi(Y|X) + k - \mathrm{negl}(n)$.*

2. *For every nonuniform, randomized polynomial-time algorithm A,*

$$\mathsf{D}_\varphi((X, Y)\|(X, A(X))) \geq k - \mathrm{negl}(n).$$

This theorem implies Theorem 1.3 as a special case, because if we take $\varphi(p) = \sum_b p_b \log p_b$, then H_φ becomes conditional Shannon entropy and D_φ becomes KL divergence.[1] Moreover, if we instead take $\varphi(p) = \|p\|_\infty := \max_b p_b$, then H_φ is equivalent (up to a monotone increasing change of variables) to average min-entropy. Unfortunately, we cannot derive Theorem 1.2 from Theorem 1.4 because this choice of φ is not differentiable, and the proof of Theorem 1.4 crucially relies on differentiability. However, it does at least give us generalizations to other entropy measures. In particular, taking $\varphi(p) = \sum_b p_b^2$ or $\varphi(p) = \sqrt{\sum_b p_b^2}$, $\mathsf{H}_\varphi(B|A)$ measures conditional forms of the

[1] In this case, $\|\nabla\varphi(p)\|_\infty$ is unbounded as p approaches the boundary of Δ_L, violating the boundedness assumption required by Theorem 1.4, but this can be addressed easily by slightly perturbing p away from the boundary (see Section 5).

collision probability (equivalently, Rényi entropy of order 2), which have proved useful in cryptography and algorithmic applications, for example in [HNO+09, CMV13]. (For interpretations of D_φ in these cases, see Section 1.6.)

A different unification of the hardness–randomness equivalences came recently in the work of Casacuberta, Dwork, and Vadhan [CDV24]. Inspired by the concurrent work of Dwork, Lee, Lin, and Tankala [DLLT23], they reinterpreted the Multicalibration Theorem from the algorithmic fairness literature [HKRR18, GKR+22] and showed how it gives a simpler proof of Theorems 1.2 and 1.3 when $\ell = O(1)$, along with other known results about average-case complexity and computational indistinguishability (namely, Impagliazzo's Hardcore Lemma [Imp95, Hol05] and the complexity-theoretic Dense Model Theorem [RTTV08]). Here is their interpretation of the Multicalibration Theorem (with some cheating, which we will correct below):

Theorem 1.5 (Multicalibration Theorem, informally stated)**.** *Let (X, Y) be a random variable distributed on $\{0,1\}^n \times \{0,1\}^\ell$ with $\ell = O(\log n)$. Then there is a function $P : \{0,1\}^n \to \{0,1\}^{O(2^\ell \cdot \log n)}$ computable by a nonuniform algorithm running in time $\mathsf{poly}(n)^{2^\ell}$ such that $(X, Y) \approx^c (X, Z)$, where for every $x \in \mathsf{Supp}(X)$, we define $Z|_{X=x}$ to be identically distributed to $Y|_{P(X)=P(x)}$.*

Notice that conditioned on the value of $P(X)$, Z and X are independent of each other. This means that, conditioned on $P(X)$, (X, Z) is the "most random" random variable of the form (X, W) that is indistinguishable from (X, Y). (Since $\ell = O(\log n)$, if (X, W) is computationally indistinguishable from (X, Y), then the marginal distribution of W must be statistically indistinguishable from Y, so the only way to noticeably increase the entropy of W given X is to make W more independent of X.)

Furthermore, when $\ell = O(1)$, P is computable in nonuniform polynomial time and outputs only $O(\log n)$ bits. In this setting, we can thus derive Theorems 1.2,1.3, and 1.4 easily. To illustrate with Theorem 1.2, consider any nonuniform polynomial-time A. Since $(X, Y) \approx^c (X, Z)$, we have:

$$\Pr[A(X) = Y] \le \Pr[A(X) = Z] + \mathrm{negl}(n) \le 2^{-\mathsf{H}_{\mathsf{avg\text{-}min}}(Z|X)} + \mathrm{negl}(n),$$

and furthermore, we can construct an A with $\Pr[A(X) = Y] = 2^{-\mathsf{H}_{\mathsf{avg\text{-}min}}(Z|X)}$, by having $A(X)$ compute $q = P(X)$ and output $b_q = \arg\max_b \Pr[Y = b | P(X) = q]$, which we can hardwire into our nonuniform polynomial-time algorithm. We can afford to hardwire all of the b_q values since P outputs $O(\log n)$ bits and hence there are only $2^{O(\log n)} = \mathsf{poly}(n)$ choices for q. Similarly, the fact that P is computable in non-uniform polynomial time and only outputs $O(\log n)$ bits implies that Z is the random variable of maximum average min-entropy given X such that $(X, Y) \approx^c (X, Z)$.

Interestingly, for Theorem 1.3 and 1.4, we get that the polynomial-time algorithm A that minimizes the divergence $\mathsf{D}_\varphi((X, Y) \| (X, A(X)))$ simply samples from Z given X (which can be done efficiently since P is polynomial-time computable and outputs only $O(\log n)$ bits). That is, the best efficiently computable predictor for Y given X (with respect to D_φ) is *identical* to the highest-H_φ distribution Z that is indistinguishable from Y given X. Furthermore, this distribution Z is the same for all of the Bregman entropy measures H_φ (as well as for $\mathsf{H}_{\mathsf{avg\text{-}min}}$).

Correcting the cheating. Above, we cheated on the order of quantifiers in the Multicalibration Theorem. P is not actually computable in time $\mathsf{poly}(n)^{2^\ell}$ for a fixed polynomial $\mathsf{poly}(\cdot)$. Rather, for every polynomial $p(n)$, there is a P computable in time $\mathsf{poly}(p(n))^{2^\ell}$, outputting $\log\left(\mathsf{poly}(p(n))^{2^\ell}\right)$ bits, such that Y and X are computationally indistinguishable from being independent conditioned

on $P(X)$, for nonuniform distinguishers that run in time $p(n)$ and with an advantage bounded by $1/p(n)$. This corrected order of quantifiers still suffices to prove Theorems 1.2, 1.3, and 1.4 when $\ell = O(1)$, without modifying those theorem statements. However, the order of quantifiers does also need to be incorporated into our informal statement that the best efficiently computable predictor for Y given X (with respect to any D_φ) is *identical* to the highest-entropy distribution Z that is indistinguishable from Y given X. (We will see precise statements later.)

In summary, the Multicalibration Theorem comes close to providing a unified picture of hardness versus randomness, but it has two significant deficiencies. First, it has a factor of 2^ℓ in the *exponent* of the runtime of P, which is unfortunately necessary (as we prove in Corollary 1.13). Second, it is only stated and used in [CDV24] for nonuniform complexity, whereas the prior proofs of Theorems 1.2, 1.3 had uniform-complexity analogues [VZ13, Zhe14].

1.2 Our Contributions

In this work, we address the aforementioned deficiencies of both Theorems 1.4 and 1.5, and thereby provide a more unified and clarified understanding of hardness versus randomness. Specifically, we provide a substantial strengthening of Theorem 1.4 with the following features in Theorem 1.10:

- The convex function φ does not need to be differentiable, and thus our generalization encompasses Theorem 1.2.
- Our theorem applies simultaneously to a set Φ of functions φ, with a complexity blow up that depends on the (nonuniform) time needed to accurately evaluate the (sub)gradients $\nabla\varphi$, for an arbitrary $\varphi \in \Phi$. For the specific φ needed to capture Theorems 1.2 and 1.3, as well as for conditional collision probability (Rényi 2-entropy), the complexity blow-up is just $\mathsf{poly}(2^\ell)$, which is affordable for $\ell = O(\log n)$. But if we take all possible functions φ, we get a blow-up that is doubly-exponential in ℓ, like obtained from the Multicalibration Theorem.
- Similarly to the Multicalibration Theorem, we deduce (subject to appropriate quantifiers as mentioned above) that the best efficiently computable prediction for Y given X (with respect to D_φ) is *identical* to the highest-H_φ distribution Z that is indistinguishable from Y given X. Furthermore, this distribution Z is the same for all of the $\varphi \in \Phi$.
- We state and prove a uniform-complexity generalization of our theorem in Theorem 1.11. This result encompasses the uniform-complexity versions of Theorems 1.2 and 1.3 as shown in [VZ12, Zhe14] and extends Theorem 1.4 to the uniform-complexity setting as well (which, to our knowledge, has not been achieved in prior work). The uniform-complexity setting plays and important role in [VZ12], which uses the uniform-complexity version of Theorem 1.3 to give refined constructions of pseudorandom generators from one-way functions.
- To justify the blow-up in our theorem depending on the complexity of (approximately) computing the (sub)gradients $\nabla\varphi$, we prove that a doubly-exponential dependence on ℓ is unavoidable if we do not place any computational efficiency assumptions on φ, even when Φ only contains a single convex function φ with bounded subgradients (Theorem 1.12). This result implies that the doubly-exponential dependence is necessary for achieving multicalibration (as well as the weaker notion of *calibrated multiaccuracy*) even when there is only a single, extremely simple distinguisher (Corollary 1.13).

A key insight in our work is that full-fledged multicalibration is overkill for proving Theorems 1.2 and 1.3. Instead, it suffices to use a combination of two substantially weaker notions, *multiaccuracy* and *weight-restricted calibration*. The latter notion (weight-restricted calibration) is the part that is tailored to the specific family Φ, imposing one constraint per function $\varphi \in \Phi$, leading to a complexity blow-up that depends on the maximum complexity of (the gradients of) the functions in Φ. See Section 1.5 for a more detailed discussion about the relationships among these notions from the algorithmic fairness literature (multicalibration, multiaccuracy, and weight-restricted calibration).

In independent works, Casacuberta, Gopalan, Kanade, and Reingold [CGKR25] and Dwork and Tankala [DT25] demonstrated the power of (variants of) a similar notion termed *calibrated multiaccuracy* (see Section 1.5 for more discussion), with Casacuberta et al. [CGKR25] giving applications in agnostic learning and to proving Impagliazzo's Hardcore Lemma (which, as mentioned earlier, is equivalent to Theorem 1.5 for the case $\ell = 1$) and Dwork and Tankala [DT25] showing that it can substitute for multicalibration in the results of [MPV24], which characterize computational indistinguishability under repeated samples.

1.3 Notation

To describe our results and techniques more precisely, it will be convenient to switch from random-variable notation as used above to a functional notion. Let L be a positive integer, which represents 2^ℓ in the discussion above. Let $\Delta_L \subseteq [0,1]^L$ be the set of probability distributions over a label space $[L] := \{1, \ldots, L\}$, where we represent discrete probability distributions by their probability mass functions. Then our random variable (X, Y) can be represented by the probability distribution μ of X and a function $g^* : \{0,1\}^n \to \Delta_L$, where $g^*(x)$ is the distribution of $Y|_{X=x}$. Similarly, we can represent another random variable Z jointly distributed with X by a *simulator* $s : \{0,1\}^n \to \Delta_L$.

We normally would describe a distinguisher d between (X, Y) and (X, Z) as a randomized function $d : \{0,1\}^n \times [L] \to \{0,1\}$. Instead, it will be convenient to describe them as functions $f : \{0,1\}^n \to [-1,1]^L$, where $f(x)_y$ represents $1 - 2\mathbb{E}[d(x,y)]$. Then computational indistinguishability between (X, Y) and (X, Z) can be conveniently described as follows.

Definition 1.6. *Let μ be a distribution on $\{0,1\}^n$ and $g^*, s : \{0,1\}^n \to \Delta_L$. We say that g^* and s are* (T, ε)-indistinguishable *with respect to μ if for every $f : \{0,1\}^n \to [-1,1]^L$ computable in nonuniform time T, we have:*

$$|\mathbb{E}_{x \sim \mu} [\langle s(x) - g^*(x), f(x) \rangle]| \leq \varepsilon,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^L$. We write $\mathsf{Ind}_\mu(g^*; T, \varepsilon)$ to denote the class of all functions $s : \{0,1\}^n \to \Delta_L$ that are (T, ε)-indistinguishable from g^*.

From now on, we fix an arbitrary distribution μ over X and will often omit it from our notations.

We define our entropy measures as follows:

Definition 1.7. *Let $\varphi : \Delta_L \to \mathbb{R}$ be a convex function. For a distribution μ on $\{0,1\}^n$ and a function $g : \{0,1\}^n \to \Delta_L$, we define*

$$\mathsf{H}_\varphi(g) = -\mathbb{E}_{x \sim \mu} [\varphi(g(x))] .$$

Consider what happens when we apply this definition to a μ and g^* representing a random variable (X, Y) on $\{0,1\}^n \times [L]$. If we take $\varphi(p) = \sum_b p_b \log p_b$, then $\mathsf{H}_\varphi(g^*) = \mathsf{H}_{\mathsf{Sh}}(Y|X)$. If we

take $\varphi(p) = \|p\|_\infty$, then $\mathsf{H}_\varphi(g^*) = -2^{-\mathsf{H}_{\text{avg-min}}(Y|X)}$, which is indeed a monotone increasing change of variables of the average min-entropy.

The divergence measures we consider are defined as follows.

Definition 1.8. *Let $\varphi : \Delta_L \to \mathbb{R}$ be a convex function. If φ is differentiable, then for $g^*, g : \{0,1\}^n \to \Delta_L$ and a distribution μ on $\{0,1\}^n$, we define the* Bregman divergence *as:*

$$\begin{aligned}
\mathsf{D}_\varphi(g^* \| g) &:= \mathbb{E}_{x\sim\mu}\left[\mathsf{D}_\varphi(g^*(x)\|g(x)\right] \\
&:= \mathbb{E}_{x\sim\mu}\left[\varphi(g^*(x)) - \varphi(g(x)) - \langle g^*(x) - g(x), \nabla\varphi(g(x))\rangle\right] \\
&= \mathsf{H}_\varphi(g) - \mathsf{H}_\varphi(g^*) - \mathbb{E}_{x\sim\mu}\left[\langle g^*(x) - g(x), \nabla\varphi(g(x))\rangle\right]
\end{aligned}$$

where $\nabla\varphi(v) \in \mathbb{R}^L$ is the gradient of φ at $v \in \Delta_L$. If φ is not differentiable, then we replace $\nabla\varphi(p)$ with a fixed choice of subgradient *of φ at each $p \in \Delta_L$. (See Section 2.1.)*

Consider what happens when we apply this definition to g^* and g representing random variables (X, Y) and $(X, A(X))$ on $\{0,1\}^n \times [L]$, respectively. If we take $\varphi(p) = \sum_b p_b \log p_b$, then $\mathsf{D}_\varphi(g^*\|g) = \mathsf{D}_{\mathsf{KL}}((X,Y)\|(X,A(X)))$. If we take $\varphi(p) = \|p\|_\infty - 1$, this function is not differentiable and its subgradient is not unique. We choose

$$\nabla\varphi(p) = -\mathbf{e}_{\arg\max_b p_b},$$

where $\mathbf{e}_b$ is the standard basis vector whose b-th coordinate is one, and $\arg\max_b v_b$ is the index of the largest coordinate of v. When there are multiple largest coordinates, we define $\arg\max_b v_b$ to be the smallest index of those largest coordinates. Now we have for $v^*, v \in \Delta_L$,

$$\begin{aligned}
\mathsf{D}_\varphi(v^*\|v) &= \|v^*\|_\infty - \|v\|_\infty - \langle v^* - v, \mathbf{e}_{\arg\max_j v_j}\rangle \\
&= \|v^*\|_\infty - \langle v^*, \mathbf{e}_{\arg\max_j v_j}\rangle \\
&= \|v^*\|_\infty - \langle v^*_{\arg\max_j v_j}\rangle.
\end{aligned}$$

Thus, we calculate the divergence between functions g^* and g as:

$$\mathsf{D}_\varphi(g^*\|g) = 2^{-\mathsf{H}_{\text{avg-min}}(Y|X)} - \Pr[Y = \arg\max_b g(X)_b].$$

Thus, minimizing $\mathsf{D}_\varphi(g^*\|g)$ over all efficiently computable g amounts to maximizing the probability that an efficiently computable $A'(x) = \arg\max_b g(x)_b$ predicts Y, aligning with Theorem 1.2.

For a time bound $T \geq 0$, we use $\mathsf{Time}(T)$ to denote the class of functions $g : \{0,1\}^n \to \Delta_L$ computable in non-uniform time T. For a function $r : \Delta_L \to \mathbb{R}^L$, we define

$$\|r\|_\infty := \sup_{p\in\Delta_L} \|r(p)\|_\infty.$$

With this notation in hand, we can restate Theorem 1.4 of Zheng as follows:

Theorem 1.9 (Characterization of generalized pseudoentropy [Zhe14])**.** *Let $\varphi : \Delta_L \to \mathbb{R}$ be a differentiable strictly convex function. Let T_φ, T'_φ be time bounds such that given a linear function Λ on Δ_L, the $p \in \Delta_L$ minimizing $\varphi(p) + \Lambda(p)$ can be computed in nonuniform time T_φ and that given*

$p \in \Delta_L$, the gradient $\nabla\varphi(p)$ can be computed in nonuniform time T'_φ. Then for every time bound $T > 0$, every $\varepsilon \in (0,1)$, there exists $T' = \mathsf{poly}(T, n, L, 1/\varepsilon) + T_\varphi$ such that for every $g^* : X \to \Delta_L$,

$$\underbrace{\max_{s \in \mathsf{Ind}(g^*;T,\varepsilon)} \mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*)}_{\textit{pseudoentropy gap}} \geq \underbrace{\min_{g \in \mathsf{Time}(T')} \mathsf{D}_\varphi(g^* \| g)}_{\textit{hardness of approximating}}. \tag{1}$$

Conversely, for every time bound T and for every $g^* : X \to \Delta_L$, we have[2]

$$\underbrace{\max_{s \in \mathsf{Ind}(g^*;T+T'_\varphi,\varepsilon)} \mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*)}_{\textit{pseudoentropy gap}} \leq \underbrace{\min_{g \in \mathsf{Time}(T)} \mathsf{D}_\varphi(g^* \| g)}_{\textit{hardness of approximating}} + \varepsilon \cdot \|\nabla\varphi\|_\infty. \tag{2}$$

Theorem 1.9 indeed implies Theorem 1.4 by the following correspondences:

$$\begin{aligned} L &= 2^\ell, \\ \mathsf{H}_\varphi(g^*) &= \mathsf{H}_\varphi(Y|X), \\ \mathsf{H}_\varphi(s) &= \mathsf{H}_\varphi(Z|X), \\ \mathsf{D}_\varphi(g^* \| g) &= \mathsf{D}_\varphi((X,Y) \| (X, A(X))). \end{aligned}$$

The assumption $\ell = O(\log n)$ in Theorem 1.4 gives $L = \mathsf{poly}(n)$. We also have $T_\varphi, T'_\varphi = \mathsf{poly}(n)$ from the assumptions in Theorem 1.4. These conditions ensure that $T' = \mathsf{poly}(T, n, 1/\varepsilon)$ in Theorem 1.9, so we can use (1) to prove that Item 2 (hardness) in Theorem 1.4 implies Item 1 (pseudoentropy). Similarly, the reverse implication follows from (2).

1.4 Formal Statement of Results

We state our main results in Theorems 1.10, 1.11, and 1.12 below.

1.4.1 Non-uniform Setting

Theorem 1.10. *Let $T > 0$ be a time bound and let $\varepsilon \in (0,1)$ be an error parameter. Let Φ be a family of convex functions $\varphi : \Delta_L \to \mathbb{R}$. Assume that given $v \in \Delta_L$, the subgradient $\nabla\varphi(v)$ is bounded in $[-1,1]^L$ and can be computed to ℓ_∞ accuracy $\varepsilon/4$ in nonuniform time T_Φ, for every $\varphi \in \Phi$. Then there exists*

$$T' = O\left(\frac{(T + T_\Phi)\log L}{\varepsilon^2} + L \cdot \mathsf{poly}(1/\varepsilon, \log L)\right) \tag{3}$$

such that for every $g^* : X \to \Delta_L$, there exists $s \in \mathsf{Ind}(g^*; T, \varepsilon) \cap \mathsf{Time}(T')$ such that

$$\mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*) \geq D_\varphi(g^* \| s) - \varepsilon \quad \textit{for every } \varphi \in \Phi. \tag{4}$$

Conversely, for every $T > 0$, there exists $T' = O(T + T_\Phi)$ such that for every $g^* : X \to \Delta_L$ and every $\varphi \in \Phi$,

$$\max_{s \in \mathsf{Ind}(g^*;T',\varepsilon/2)} (\mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*)) \leq \min_{g \in \mathsf{Time}(T)} \mathsf{D}_\varphi(g^* \| g) + \varepsilon. \tag{5}$$

[2]The dependency on $\|\nabla\varphi\|_\infty$ in (2) ensures that (2) remains invariant after scaling φ by any positive factor.

In Theorem 1.10, the *single* simulator s satisfies (4) for *all* $\varphi \in \Phi$, and it simultaneously achieves indistinguishability ($s \in \mathsf{Ind}(g^*; T, \varepsilon)$) and computational efficiency ($s \in \mathsf{Time}(T')$). Therefore, (4) is stronger than the following guarantee akin to (1) in Theorem 1.9:

$$\max_{s \in \mathsf{Ind}(g^*;T,\varepsilon)} \mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*) \geq \min_{g \in \mathsf{Time}(T')} D_\varphi(g^* \| g) - \varepsilon \quad \text{for every } \varphi \in \Phi.$$

Here, the entropy maximizer s on the left (the *simulator*) and the divergence minimizer g on the right (the *approximator*) can be different and can change based on the choice of φ. By contrast, in (4), the single function s *simultaneously* plays the roles of the simulator and the approximator *simultaneously* for all $\varphi \in \Phi$. This gives a rigorous confirmation for the aforementioned informal intuition from the Multicalibration Theorem (Theorem 1.5) that there is a close correspondence between the best efficiently computable approximator g for g^* (with respect to any D_φ) and the highest-entropy simulator s that is indistinguishable from g^*.

The increase in complexity at (3) in Theorem 1.10 is polynomial in $L = 2^\ell$, a significant improvement over the doubly exponential dependence on ℓ in Theorem 1.5. We achieve this by focusing on a class Φ of functions φ whose subgradients $\nabla\varphi$ are efficiently computable (up to small ℓ_∞ error). This efficiency assumption is naturally satisfied by common entropy notions such as the Shannon entropy ($\varphi(v) = \sum_i v_i \ln v_i$), the min entropy ($\varphi(v) = \|v\|_\infty$), and the collision probability ($\varphi(v) = \sum_i v_i^2$, see Section 1.6). One caveat is that Theorem 1.10 also requires the subgradient $\nabla\varphi(v)$ to be bounded, which is not satisfied by the Shannon entropy when v is close to the boundary of the probability simplex Δ_L. However, this turns out to be a minor issue that can be easily addressed by slightly perturbing v away from the boundary of Δ_L (see Section 5).

In Theorem 1.12 (which we present shortly), we show that some efficiency assumption on the convex functions φ is necessary to avoid an exponential dependence on L (i.e. to avoid a doubly-exponential dependence on ℓ).

In contrast to Theorem 1.9, our Theorem 1.10 does not require φ to be differentiable or strictly convex. When φ is not differentiable, its subgradient at some $v \in \Delta_L$ may not be unique, and Theorem 1.10 holds as long as we use any fixed choice of subgradient to define the Bregman divergence D_φ (assuming that the subgradient is bounded and can be computed up to small ℓ_∞ error in time T). Therefore, our Theorem 1.10 encompasses both Theorems 1.2 and 1.3.

By (5), the function $s \in \mathsf{Ind}(g^*; T, \varepsilon) \cap \mathsf{Time}(T')$ in (4) satisfies

$$\mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*) \geq \max_{s' \in \mathsf{Ind}(g^*;T'',\varepsilon)} (\mathsf{H}_\varphi(s') - \mathsf{H}_\varphi(g^*)) - 2\varepsilon \quad \text{for every } \varphi \in \Phi,$$

where

$$T'' = O(T' + T_\Phi) = O(T') = O\left(\frac{(T + T_\Phi)\log L}{\varepsilon^2} + L \cdot \mathsf{poly}(1/\varepsilon, \log L)\right).$$

This means that the *single* indistinguishable function $s \in \mathsf{Ind}(g^*; T, \varepsilon)$ achieves comparable or higher entropy than *every* $s' \in \mathsf{Ind}(g^*; T'', \varepsilon)$ w.r.t. *every* entropy notion H_φ for $\varphi \in \Phi$. In this sense, s is a *universal simulator* for the pseudoentropy of g^*.

Combining the two directions (4) and (5) of Theorem 1.10 in a different order, we get that for every time bound $T > 0$ and every $g^* : \{0,1\}^n \to \Delta_L$, there exist $T' = O(T + T_\Phi)$,

$$T'' = O\left(\frac{(T' + T_\Phi)\log L}{\varepsilon^2} + L \cdot \mathsf{poly}(1/\varepsilon, \log L)\right)$$

and a function $s \in \mathsf{Ind}(g^*; T', \varepsilon) \cap \mathsf{Time}(T'')$ such that

$$\mathsf{D}_\varphi(g^* \| s) \le \min_{g \in \mathsf{Time}(T)} \mathsf{D}_\varphi(g^* \| g) + 2\varepsilon \quad \text{for every } \varphi \in \Phi.$$

This means that the *single* low-complexity function $s \in \mathsf{Time}(T'')$ achieves comparable or better approximation error than *every* $g \in \mathsf{Time}(T)$ w.r.t. *every* divergence notion D_φ for $\varphi \in \Phi$. In the recent learning theory literature, a function s satisfying this property is termed an *omnipredictor* [GKR+22]. Our Theorem 1.10 thus recovers a result of [GHK+23] showing a construction of omnipredictors based on *calibrated multiaccuracy*. (See Section 1.5 for more discussion.)

1.4.2 Uniform Setting

In Theorem 1.11 below, we prove a uniform version of Theorem 1.10 that encompasses the uniform versions of Theorems 1.2 and 1.3 in [VZ12, Zhe14] and extends Theorem 1.4 also to the uniform setting (which has not been explicitly achieved in prior work). In the uniform setting, instead of having a single function $g^* : \{0,1\}^n \to [L]$, we are interested in a family of functions $g^*_{n,L} : \{0,1\}^n \to [L]$ defined for varying choices of input length n and label space size L. Theorem 1.10 only allows us to construct an efficiently computable function s satisfying (4) separately for each choice of (n, L), so it does not guarantee the existence of a single uniform algorithm (e.g. a Turing machine) that efficiently computes s for all choices of (n, L). Our Theorem 1.11 addresses this limitation.

To state Theorem 1.11, we use terminology similar to the uniform versions of Theorems 1.2 and 1.3 in [VZ12, Zhe14], where we formalize a uniform distinguisher as an efficient uniform oracle $\mathcal{A}$ that, when given sample access to $g^*, g : \{0,1\}^n \to \Delta_L$ on a distribution μ,[3] aims to output a function $f : \{0,1\}^n \to [-1,1]^L$ that witnesses the violation of Definition 1.6:

$$\mathbb{E}_{x \sim \mu} \langle g(x) - g^*(x), f(x) \rangle > \varepsilon. \tag{6}$$

We say g is $(\mathcal{A}, \varepsilon)$-*distinguishable* from g^* if (6) holds with high probability, and otherwise we say g is $(\mathcal{A}, \varepsilon)$-indistinguishable. We defer the formal definition to Definitions 3.6 and 3.8.

Our goal is to construct an $(\mathcal{A}, \varepsilon)$-indistinguishable function s satisfying (4) using an efficient uniform algorithm that takes $\mathcal{A}$ as a subroutine, and we still want s to satisfy (4) simultaneously for all $\varphi \in \Phi$. To achieve this, we assume that the function class $\{\nabla\varphi\}_{\varphi \in \Phi}$ is *weakly-agnostically learnable*, which is trivially satisfied when Φ only contains a single φ whose subgradient $\nabla\varphi$ can be efficiently approximated (as in Theorems 1.2 and 1.3). Roughly speaking, we assume that there is a weak-agnostic learner $\mathcal{B}$ that, when given i.i.d. data points $(v_1, z_1), \ldots, (v_m, z_m) \in \Delta_L \times [-1,1]^L$ drawn from an unknown distribution μ', can efficiently find $r : \Delta_L \to [-1,1]$ such that $\mathbb{E}_{(v,z) \sim \mu'} \langle z, r(v) \rangle$ is comparable with or larger than $\sup_{\varphi \in \Phi} \mathbb{E}_{(v,z) \sim \mu'} \langle z, \nabla\varphi(v) \rangle$. We defer the formal definition to Definition 4.1.

Theorem 1.11 (Uniform version of Theorem 1.10; informal statement of Theorem 4.3)**.** *Let Φ be a class of convex functions $\varphi : \Delta_L \to \mathbb{R}$, and let μ be an arbitrary distribution on $\{0,1\}^n$. Let $\mathcal{A}$ be an arbitrary distinguishing oracle, and let $\mathcal{B}$ be an ε_1-weak agnostic learner for $\{\nabla\varphi\}_{\varphi \in \Phi}$. We can use $\mathcal{A}$ and $\mathcal{B}$ as sub-routines to construct an efficient (uniform) algorithm $\mathcal{S}$ that, given sample access to $g^* : \{0,1\}^n \to \Delta_L$ on μ, with high probability outputs a (low-complexity) function s that is $(\mathcal{A}, \varepsilon)$-indistinguishable from g^*, and such that*

$$\mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*) \ge D_\varphi(g^* \| s) - \varepsilon_1 \quad \textit{for every } \varphi \in \Phi.$$

[3] This means that $\mathcal{A}$ can obtain random samples $(x, y, y^*) \in \{0,1\}^n \times [L] \times [L]$, where we first draw x from μ and then independently draw $y \sim g(x), y^* \sim g^*(x)$.

Theorem 1.11 encompasses the uniform versions of Theorems 1.2 and 1.3 in [VZ12, Zhe14] by choosing the entropy notion H_φ accordingly. In the uniform versions of Theorems 1.2 and 1.3, we assume that every function s' with $\mathsf{H}_\varphi(s') - \mathsf{H}_\varphi(g^*)$ exceeding some threshold β is $(\mathcal{A}, \varepsilon)$-distinguishable from g^*. Under this assumption, since the function s from Theorem 1.11 is $(\mathcal{A}, \varepsilon)$-indistinguishable from g^*, it must hold that $\mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*) \le \beta$, which implies that $\mathsf{D}_\varphi(g^* \| s) \le \beta + \varepsilon_1$. In this case, the algorithm $\mathcal{S}$ from Theorem 1.11 becomes an efficient uniform algorithm that approximates g^* within low D_φ error, exactly as needed to prove the uniform versions of Theorems 1.2 and 1.3. We remark that the uniform versions of Theorems 1.2 and 1.3 in [VZ12, Zhe14] also have the reverse direction that can also be extended to general entropy notions H_φ to get a uniform version of the reverse direction (5) of Theorem 1.10. We omit this result as this generalization can be proved straightforwardly using the same proof idea as its non-uniform counterpart.

1.4.3 Exponential Lower Bound

In our Theorem 1.10, we make a computational efficiency assumption about the convex functions $\varphi \in \Phi$ corresponding to the entropy notions considered in the theorem: we assume that the subgradient function $\nabla\varphi$ can be approximated by a circuit of size at most T_Φ for every $\varphi \in \Phi$. Correspondingly, the circuit complexity of the function s guaranteed by the theorem has circuit complexity depending polynomially on T_Φ, in addition to the polynomial dependency on T, L and $1/\varepsilon$.

In this section, we prove the following lower-bound theorem showing that some complexity assumption on φ is necessary to avoid an exponential dependency on L in the circuit complexity of s. This lower bound holds even when Φ only contains a single convex function φ and there is only a single, extremely simple distinguisher f:

Theorem 1.12. *For every sufficiently large positive integer n, choosing $L = n$, there exist a distribution μ over $\{0,1\}^n$, a function $g^* : \{0,1\}^n \to \Delta_L$, an $O(n)$-sized circuit $f : \{0,1\}^n \to [-1,1]^L$, and a convex function $\varphi : \Delta_L \to \mathbb{R}$ with bounded subgradient $\nabla\varphi(v) \in [-1,1]^L$ for every $v \in \Delta_L$ that satisfy the following property. Let $s : \{0,1\}^n \to \Delta_L$ be an arbitrary function that is $(\{f\}, 0.05)$-indistinguishable from g^* (i.e. $s \in \mathsf{Ind}(g^*; \{f\}, 0.05)$) and satisfies the following inequality:*

$$\mathsf{H}_\varphi(s) \ge \mathsf{H}_\varphi(g^*) - 0.05. \tag{7}$$

Then s must have circuit complexity $\exp(\Omega(n)) = \exp(\Omega(L))$. Moreover, we can simply choose f to be the identity function: $f(x) = x \in \{0,1\}^n \subseteq [-1,1]^L$ for every $x \in \{0,1\}^n$.

We would like to remark that the condition (7) is a very mild assumption. In particular, by the nonnegativity of Bregman divergences, (7) is a necessary condition for the following guarantee of our main theorem (Theorem 1.10):

$$\mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*) \ge D_\varphi(g^* \| s) - 0.05. \tag{8}$$

Therefore, Theorem 1.12 implies an exponential lower bound on the circuit complexity of every function $s \in \mathsf{Ind}(g^*; \{f\}, 0.05)$ that satisfies the guarantee (4) of Theorem 1.10, even when Φ only contains a single function φ and there is only a single, extremely simple distinguisher f (the identity function).

There is no contradiction between the $\exp(\Omega(L))$ lower bound (Theorem 1.12) and the $\mathsf{poly}(L)$ upper bound (Theorem 1.10) on the circuit complexity of s because the upper bound assumes that the convex function φ has low complexity (specifically, its subgradient can be approximated by a

circuit of size T_Φ). Thus, Theorem 1.12 shows that some assumptions on the complexity of φ are necessary for Theorem 1.10 to hold, even when Φ only contains a single convex function φ.

Our proof of Theorem 1.12 uses coding-theoretic ideas (specifically, the existence of a sufficiently large *design*) as well as a probabilistic counting argument. See Section 6 for the formal proof.

As a corollary of Theorem 1.12, we give an exponential lower bound on the circuit complexity for achieving multiaccuracy (i.e. indistinguishability) plus weight-restricted calibration, even for a single, extremely simple distinguisher f (the identity function) and a single weight function $\nabla\varphi : \Delta_L \to [-1,1]^L$. Consequently, this lower bound also holds for stronger notions such as calibrated multiaccuracy and multicalibration. See Section 1.5 for a more detailed discussion about these notions from the algorithmic fairness literature (multiaccuracy, weight-restricted calibration, calibrated multiaccuracy, and multicalibration).

Corollary 1.13. *For every sufficiently large positive integer n, choosing $L = n$, the same distribution μ, function g^*, distinguisher f and convex function φ from Theorem 1.12 have the following property. Let $s : \{0,1\}^n \to \Delta_L$ be an arbitrary function satisfying the following two conditions:*

$$|\mathbb{E}_{x\sim\mu}\langle s(x) - g^*(x), f(x)\rangle| \le 0.05, \qquad \textbf{(Indistinguishability, a.k.a. multiaccuracy)}$$
$$\mathbb{E}_{x\sim\mu}\langle s(x) - g^*(x), \nabla\varphi(s(x))\rangle \le 0.05. \qquad \textbf{(Weight-restricted calibration)}$$

Then s must have circuit complexity $\exp(\Omega(n)) = \exp(\Omega(L))$.

1.5 Proof Idea: Regularity Lemma with Weight-Restricted Calibration

We now give a high-level description of our proof idea for the first part of Theorem 1.10, i.e. the forward direction (4). Our proof of the reverse direction (5) follows the same ideas as in prior work [VZ12, Zhe14].

Our idea is very simple in hindsight. As mentioned earlier, a key difference between our guarantee (4) from previous ones (e.g. (1)) is that the same function s appears on both sides of the inequality, playing the roles of the simulator and the approximator simultaneously. This makes it natural to calculate the diference between the two sides. Specifically, using the definitions of H_φ and D_φ in Definitions 1.7 and 1.8, we obtain the following identity, which is the central technical insight of our work:

$$\underbrace{\mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*)}_{\text{pseudoentropy gap}} - \underbrace{\mathsf{D}_\varphi(g^*\|s)}_{\text{hardness of approximating}} = \langle g^* - s, \nabla\varphi \circ s\rangle. \tag{9}$$

In the right-hand side of (9) and for the rest of the paper, for any two functions $g, h : \{0,1\}^n \to \mathbb{R}^L$, we define their inner product as

$$\langle g, h\rangle := \mathbb{E}_{x\sim\mu}\langle g(x), h(x)\rangle,$$

where μ is the distribution on $\{0,1\}^n$ we fix in advance.

By (9), our goal (4) simplifies to constructing s satisfying

$$\langle s - g^*, \nabla\varphi \circ s\rangle \le \varepsilon \quad \text{for every } \varphi \in \Phi, \tag{10}$$

such that s is efficiently computable and indistinguishable from g^*. In Theorem 1.10, we assume that $\nabla\varphi$ is bounded in $[-1,1]^L$ and can be approximated within ℓ_∞ error $\varepsilon/4$ in non-uniform time

T_Φ. Thus for every $\varphi \in \Phi$, there exists $r_\varphi : \Delta_L \to [-1,1]$ computable in non-uniform time T_Φ such that $\|r_\varphi - \nabla\varphi\|_\infty \le \varepsilon/4$. This gives a sufficient condition for (10):

$$\langle s - g^*, r_\varphi \circ s\rangle \le \varepsilon/2 \quad \text{for every } \varphi \in \Phi. \quad \textbf{(Weight-restricted calibration)} \tag{11}$$

As we will explain shortly, the condition (11) is termed *weight-restricted calibration* in the recent literature on algorithmic fairness. On the other hand, the requirement that s is indistinguishable from g^* (Definition 1.6) amounts to

$$|\langle s - g^*, f\rangle| \le \varepsilon \quad \text{for every } f \in \mathcal{F}, \quad \textbf{(Indistinguishability, a.k.a. multiaccuracy)} \tag{12}$$

where $\mathcal{F}$ denotes the class of functions $f : \{0,1\}^n \to [-1,1]^L$ computable in non-uniform time T. Again, as we will explain below, the indistinguishability condition (11) is termed *multiaccuracy* in the recent literature on algorithmic fairness.

The first half of Theorem 1.10 requires us to find a function $s \in \mathsf{Ind}(g^*; T, \varepsilon) \cap \mathsf{Time}(T')$ satisfying (4). Based on the our discussion above, the requirement $s \in \mathsf{Ind}(g^*; T, \varepsilon)$ is equivalent to (12), and a sufficient condition for (4) is given by (11). Thus our goal becomes finding $s \in \mathsf{Time}(T')$ satisfying both (11) and (12).

The structure of (11) and (12) points us to previous ideas about *regularity lemmas*. Specifically, the Complexity-Theoretic Regularity Lemma of Trevisan, Tulsiani, and Vadhan [TTV09] and its extension to $L > 2$ in the Leakage Simulation Lemma of Jetchev and Pietrzak [JP14], allows us to construct $s \in \mathsf{Time}(T')$ that satisfies (12) alone. This result can be proved using a boosting algorithm. We start with a simple $s : \{0,1\}^n \to \Delta_L$ (e.g. $s(x)$ is the uniform distribution over $[L]$ for every $x \in \{0,1\}^n$). Whenever s does not satisfy (12), there exists a (low-complexity) $f \in \mathcal{F}$ that witnesses this violation, which we use to update s to reduce a nonnegative potential function by a significant amount. By the nonnegativity of the potential function, there can only be a bounded number of updates, after which s has to satisfy (12). Since each $f \in \mathcal{F}$ has non-uniform complexity at most T, each update increases the complexity of s only by a bounded amount, and since the number of total updates is also bounded, we can ensure that the final s belongs to $\mathsf{Time}(T')$. Following the work of Vadhan and Zheng [VZ13] which proves a uniform version of the regularity lemma, we apply the classic Multiplicative Weights algorithm (a special case of Mirror Descent) to extend the boosting idea from [TTV09] to general L (see Sections 2.2 and 2.3).

A limitation of the complexity-theoretic regularity lemmas is that they only establish (12), but we need (11) and (12) simultaneously. The two conditions (11) and (12) look very similar, with both r_φ in (11) and f in (12) having bounded non-uniform complexity. The difference is that (11) is *circular*: the "distinguishing function" $r_\varphi \circ s$ that s tries to "fool" depends on the function s itself.

A key insight from the recent algorithmic fairness literature about *multicalibration* [HKRR18, GKSZ22] is that the circularity in (11) does not incur any new major technical challenge: the exact same boosting idea from the Regularity Lemma allows us to construct $s \in \mathsf{Time}(T')$ satisfying both (11) and (12), which is exactly what we need to establish Theorem 1.10. We view condition (11) as an instantiation of *weight-restricted calibration*, as introduced by Gopalan, Kim, Singhal, and Zhao [GKSZ22].[4] This is a relaxed version of the following condition called *calibration*:

$$\mathbb{E}_{x\sim\mu}[s(x) - g^*(x) | s(x)] \approx \mathbf{0}, \quad \textbf{(Calibration)} \tag{13}$$

[4]They called it *weighted calibration* (see also [GHR24]). We modify the terminology to more clearly convey that it is a weaker condition than calibration.

where $\mathbf{0} \in \mathbb{R}^L$ is the zero vector. Indeed, calibration (13) is equivalent to

$$\langle s - g^*, r \circ s \rangle \approx 0 \quad \text{for } \textit{all } r : \Delta_L \to [-1,1]^L, \quad \textbf{(Equivalent def. of calibration)} \tag{14}$$

where each $r : \Delta_L \to [-1,1]^L$ is termed a *weight function* [GKSZ22]. The weight-restricted calibration condition (11) is a relaxation of (full) calibration (14) in that it only considers a restricted family of weight functions $\{r_\varphi\}_{\varphi \in \Phi}$ instead of all weight functions (and it replaces "≈ 0" with "$\leq \varepsilon$").

As pointed out by Dwork, Lee, Lin, and Tankala [DLLT23], the indistinguishability condition (12) (previously studied as in the Complexity-Theoretic Regularity Lemma [TTV09]) was rediscovered in the algorithmic fairness literature under the term *multiaccuracy* [HKRR18, KGZ19] and it is natural to ask whether stronger notions from the algorithmic fairness literature have implications in complexity theory and other domains where similar regularity lemmas are used (such as graph theory and cryptography). This question has been explored by many works in recent research [HLNR23, DLLT23, CDV24]. Our work (especially the observation (9)) demonstrates the power of adding weight-restricted calibration (11) to the standard indistinguishability / multiaccuracy condition (12) guaranteed by the regularity lemma for generalizing and strenthening pseudoentropy characterizations.

The combination of multiaccuracy (12) and (full) calibration (13) (or (14)) is termed *calibrated multiaccuracy* in [GHK$^+$23]. That work uses calibrated multiaccuracy to achieve omniprediction more efficiently than the *multicalibration*-based approach of [GKR$^+$22] (we discuss multicalibration in the next paragraph). The notion of multiaccuracy (12) plus weight-restricted calibration (11) (as we use in our work) is weaker than calibrated multiaccuracy and suffices to ensure omniprediction in many cases, leading to further improved sample complexity and computational efficiency even when $L = 2$ [HTY25, OKK25]. Independent work [CGKR25, DT25] demonstrates the power of (variants of) calibrated multiaccuracy for other cryptographic, complexity-theoretic, and learning-theoretic applications (as we discussed in Section 1.2).

As mentioned earlier, the work of Casacuberta, Dwork, and Vadhan [CDV24] uses *multicalibration* to establish a characterization for pseudo-min-entropy and pseudo-Shannon-entropy. Introduced by [HKRR18], multicalibration is the following much stronger condition than (11), (12) and (13) combined:

$$\mathbb{E}_{x \sim \mu}[\langle s(x) - g^*(x), f(x) \rangle | s(x)] \approx 0 \quad \text{for every } f \in \mathcal{F}. \quad \textbf{(Multicalibration)}$$

A contribution of our work is to show that multicalibration is an overkill for pseudoentropy characterizations and leads to the unnecessary doubly exponential dependence on ℓ in Theorem 1.5 (our Corollary 1.13 shows that this doubly exponential dependence cannot be avoided for multicalibration and even for the weaker notion of calibrated multiaccuracy). Based on our observation (9), the much weaker condition of multiaccuracy (12) plus weight-restricted calibration (11) is sufficient even for more general entropy notions H_φ, giving the improved polynomial dependence on $L = 2^\ell$ in Theorem 1.10.

1.6 Examples beyond Shannon and Min Entropy

Our pseudoentropy characterization (Theorem 1.10) holds for a general family Φ of entropy notions. Here we give two examples of entropy notions beyond Shannon and min entropy.

Consider the following choice of φ with its corresponding Bregman divergence:

$$\varphi(v) = \|v\|_2^2 = \sum_{i \in [L]} v_i^2, \quad \mathsf{D}_\varphi(u \| v) = \|u - v\|_2^2 = \|u\|_2^2 + \|v\|_2^2 - 2\langle u, v \rangle. \tag{15}$$

To interpret the entropy and divergence notions from this choice of φ, let us consider a pair of jointly distributed random variables $(X, Y) \in \{0,1\}^n \times [L]$ with X drawn from a distribution μ over $\{0,1\}^n$, and define $g^* : \{0,1\}^n \to \Delta_L$ so that $g^*(x)$ is the conditional distribution of Y given $X = x$. In this case, the entropy notion H_φ is the negative conditional collision probability $\mathsf{CP}(g^*(X)|X)$ used, for example, in [CMV13] for the analysis of hash functions:

$$\mathsf{H}_\varphi(g^*) = -\mathbb{E}_{x\sim\mu}[\varphi(g^*(x))] = -\mathsf{CP}(g^*(X)|X).$$

Here,

$$\mathsf{CP}(g^*(X)|X) := \Pr[Y' = Y],$$

where (X, Y') is identically distributed as (X, Y), and Y' is conditionally independent from Y given X.

By (15), for a fixed $g^* : \{0,1\}^n \to \Delta_L$, minimizing the Bregman divergence $\mathsf{D}_\varphi(g^*\|g)$ over $g : \{0,1\}^n \to \Delta_L$ corresponds to maximizing

$$2\langle g^*, g\rangle - \mathbb{E}_{x\sim\mu}\|g(x)\|_2^2 = 2\Pr[g^*(X) = g(X)] - \mathsf{CP}(g(X)|X), \tag{16}$$

where the first term $\Pr[g^*(X) = g(X)]$ is the prediction accuracy of g w.r.t. g^* (the same as in Theorem 1.2):

$$\Pr[g^*(X) = g(X)] := \Pr[Y = \hat{Y}],$$

where $X \sim \mu$, and conditioned on X, we independently draw $Y \sim g^*(X), \hat{Y} \sim g(X)$. The second term on the right-hand side of (16) can be viewed as a regularizer that encourages g to have lower collision probability (and thus higher entropy H_φ). Our characterization Theorem 1.10 implies that the best that an efficient g can do for this regularized optimization task is equivalent to the conditional pseudo-collision-probability of $g^*(X)$ given X (i.e., the pseudo / computational analogue of $\mathsf{CP}(g^*(X)|X)$).

Similarly, we can take

$$\varphi(v) = \|v\|_2 = \sqrt{\sum_i v_i^2}, \quad \mathsf{D}_\varphi(u\|v) = \frac{\|u\|_2\|v\|_2 - \langle u, v\rangle}{\|v\|_2}$$

to get an equivalence between maximzing the following objective over efficient g

$$\Pr[g(X) = g^*(X)]/\mathsf{CP}^{1/2}(g(X)|X),$$

and the conditional pseudo-square-root-collision-probability of $g^*(X)$ given X (i.e., the pseudo / computational analogue of $\mathsf{CP}^{1/2}(g^*(X)|X)$ used, for example, in [HNO+09] to construct statistically hiding commitments and statistical zero-knowledge arguments).

2 Preliminaries

For simplicity, we consider general and abstract vectors $g, h, f \in \mathbb{R}^L$ in this section. Later we will instantiate the vector g here as the function value $g(x) \in \Delta_L \subseteq \mathbb{R}^L$ (as in Definition 1.6) assigned to a specific input $x \in \{0,1\}^n$, and similarly instantiate the vector f here as the function value $f(x) \in [-1,1]^L$ of a distinguisher $f : \{0,1\}^n \to [-1,1]^L$ on a specific input $x \in \{0,1\}^n$.

2.1 Convex Analysis Basics

Let $\mathcal{G} \subseteq \mathbb{R}^L$ be a non-empty bounded convex set. Let $\varphi : \mathcal{G} \to \mathbb{R}$ be a convex function on $\mathcal{G}$. We say $h \in \mathbb{R}^L$ is a *subgradient* of φ at $g \in \mathcal{G}$ if

$$\varphi(g') - \varphi(g) \geq \langle g' - g, h \rangle \quad \text{for every } g' \in \mathcal{G},$$

or equivalently,

$$\langle g, h \rangle - \varphi(g) = \max_{g' \in \mathcal{G}} (\langle g', h \rangle - \varphi(g')). \tag{17}$$

The *convex conjugate* of φ, denoted by $\psi : \mathbb{R}^L \to \mathbb{R}$, is defined as follows:

$$\psi(h) := \sup_{g \in \mathcal{G}} (\langle g, h \rangle - \varphi(g)) \quad \text{for every } h \in \mathbb{R}^L. \tag{18}$$

By the boundedness of $\mathcal{G}$ and the convexity of φ, the supremum above is always well-defined (i.e., it is neither $+\infty$ nor $-\infty$). It is easy to verify that ψ is convex and closed.[5] The *Fenchel-Young divergence* $\Gamma_{\varphi,\psi} : \mathcal{G} \times \mathbb{R}^L \to \mathbb{R}$ is defined as follows:

$$\Gamma_{\varphi,\psi}(g, h) := \varphi(g) + \psi(h) - \langle g, h \rangle \geq 0 \quad \text{for every } g \in \mathcal{G} \text{ and } h \in \mathbb{R}^L.$$

The nonnegativity of the divergence follows immediately from the definition of ψ in (18). We will often omit the subscripts and simply write Γ in place of $\Gamma_{\varphi,\psi}$. By the characterization of subgradients in (17), for any $g \in \mathcal{G}$ and $h \in \mathbb{R}^L$,

$$h \text{ is a subgradient of } \varphi \text{ at } g \iff \Gamma(g, h) = 0 \implies g \text{ is a subgradient of } \psi \text{ at } h. \tag{19}$$

Note that the converse of the second implication is not generally true when φ is discontinuous at the boundary of $\mathcal{G}$. We say a pair $(g, h) \in \mathcal{G} \times \mathbb{R}^L$ is a *conjugate pair* w.r.t. (φ, ψ) if $\Gamma_{\varphi,\psi}(g, h) = 0$.

The sub-gradient of φ at any $g \in \mathcal{G}$ may not be unique (and it may not exist when g is at the boundary of $\mathcal{G}$). When we have a pre-determined choice $\nabla\varphi(g)$ among the subgradients, we define the *Bregman divergence* $D_\varphi(g' \| g)$ as follows:

$$\begin{aligned} D_\varphi(g' \| g) := \Gamma(g', \nabla\varphi(g)) &= \varphi(g') + \psi(\nabla\varphi(g)) - \langle g', \nabla\varphi(g) \rangle \\ &= \varphi(g') - \varphi(g) - \langle g' - g, \nabla\varphi(g) \rangle, \end{aligned}$$

where the last equation uses the fact that $(g, \nabla\varphi(g))$ is a conjugate pair (by the equivalence in (19)):

$$0 = \Gamma(g, \nabla\varphi(g)) = \varphi(g) + \psi(\nabla\varphi(g)) - \langle g, \nabla\varphi(g) \rangle.$$

If we additionally assume that φ is closed (which holds, for example, when φ is continuous on a closed domain $\mathcal{G}$), the supremum in (18) can always be attained by some $g \in \mathcal{G}$. Moreover, it is a standard result that for a closed convex function φ, taking convex conjugate twice gives back the same function. That is, the closedness assumption of φ ensures that φ and ψ are convex conjugates of each other:

$$\psi(h) = \max_{g \in \mathcal{G}} (\langle g, h \rangle - \varphi(g)) \quad \text{for every } h \in \mathbb{R}^L, \tag{20}$$

[5] As standard terminology in convex analysis, we say ψ is *closed* if its epigraph $\{(h, y) \in \mathbb{R}^L \times \mathbb{R} : y \geq \psi(h)\}$ is closed (see e.g. [Roc97]).

$$\varphi(g) = \sup_{h \in \mathbb{R}^L} \left(\langle g, h \rangle - \psi(h) \right) \quad \text{for every } g \in \mathcal{G}. \tag{21}$$

In this case, by the characterization of subgradients in (17), for any $g \in \mathcal{G}$ and $h \in \mathbb{R}^L$, the following three statements are equivalent:

1. $\Gamma(g, h) = 0$, i.e., (g, h) is a conjugate pair;
2. h is a subgradient of φ at g;
3. g is a subgradient of ψ at h.

2.2 Mirror Descent

As in the previous section, we consider a non-empty bounded convex set $\mathcal{G} \subseteq \mathbb{R}^L$ and a convex function φ defined on it, with ψ being the convex conjugate of φ. Now we additionally assume that ψ is smooth:

Definition 2.1 (Smoothness)**.** *For $\lambda \geq 0$ and a norm $\|\cdot\|$ on $\mathbb{R}^L$, we say a convex function $\psi : \mathbb{R}^L \to \mathbb{R}$ is* λ-smooth *w.r.t. $\|\cdot\|$ if for every $h_0, h \in \mathbb{R}^L$, letting g_0 be a subgradient of ψ at h_0, we have*

$$\psi(h) \leq \psi(h_0) + \langle g_0, h - h_0 \rangle + \frac{\lambda}{2} \|h - h_0\|^2. \tag{22}$$

If a convex function $\psi : \mathbb{R}^L \to \mathbb{R}$ is λ-smooth, it is necessarily differentiable with the *subgradient* g_0 being the *gradient* at h_0, implying the uniqueness of the subgradient. We can thus uniquely define $\nabla\psi(h_0) := g_0$.

We additionally assume that φ is a closed function on the bounded domain $\mathcal{G}$ (and recall that ψ is the convex conjugate of φ). This ensures that $\nabla\psi(h) \in \mathcal{G}$ for every $h \in \mathbb{R}^L$ because the maximizer $g \in \mathcal{G}$ in (20) achieves $\Gamma(g, h) = 0$, and thus $\nabla\psi(h) = g \in \mathcal{G}$.

The Mirror Descent algorithm runs in iterations $k = 1, \ldots, K$, where each iteration k updates a conjugate pair $(g_k, h_k) \in \mathcal{G} \times \mathbb{R}^L$ to a new conjugate pair (g_{k+1}, h_{k+1}) given a signal vector $f \in \mathbb{R}^L$ as follows:

$$\begin{aligned} h_{k+1} &\leftarrow h_k - f; \\ g_{k+1} &\leftarrow \nabla\psi(h_{k+1}). \end{aligned} \tag{23}$$

The following standard lemma is central to analyzing mirror descent. It allows us to track the progress of the update from (g_k, h_k) to (g_{k+1}, h_{k+1}) using the nonnegative quantity $\Gamma(g^*, h_k)$ as a potential function for an arbitrary $g^* \in \mathcal{G}$. The lemma shows that the potential function $\Gamma(g^*, h_k)$ must significantly decrease if $\langle g_k - g^*, f \rangle$ is large relative to $\|f\|^2$.

Lemma 2.2 (Mirror Descent)**.** *Let $\varphi : \mathcal{G} \to \mathbb{R}$ be a closed convex function on a bounded convex set $\mathcal{G} \subseteq \mathbb{R}^L$, and let $\psi : \mathbb{R}^L \to \mathbb{R}$ be its convex conjugate. Assume that ψ is λ-smooth w.r.t. norm $\|\cdot\|$ for some $\lambda \geq 0$. Assume conjugate pairs $(g_k, h_k), (g_{k+1}, h_{k+1}) \in \mathcal{G} \times \mathbb{R}^L$ satisfy* (23) *for some $f \in \mathbb{R}^L$. Then for every $g^* \in \mathcal{G}$,*

$$\Gamma(g^*, h_k) - \Gamma(g^*, h_{k+1}) \geq \langle g_k - g^*, f \rangle - \frac{\lambda}{2} \|f\|^2. \tag{24}$$

Proof. We start by simplifying (24) using the following equations:

$$\begin{aligned} f &= h_k - h_{k+1}, \\ \Gamma(g^*, h_k) &= \varphi(g^*) + \psi(h_k) - \langle g^*, h_k \rangle, \\ \Gamma(g^*, h_{k+1}) &= \varphi(g^*) + \psi(h_{k+1}) - \langle g^*, h_{k+1} \rangle. \end{aligned}$$

Our goal (24) simplifies to

$$\psi(h_k) - \psi(h_{k+1}) \geq \langle g_k, h_k - h_{k+1} \rangle - \frac{\lambda}{2} \|h_k - h_{k+1}\|^2.$$

Since (g_k, h_k) is a conjugate pair, we have $g_k = \nabla\psi(h_k)$. The inequality above then follows immediately from (22) by setting $(g_0, h_0) = (g_k, h_k), h = h_{k+1}$. □

2.3 The Multiplicative Weights Algorithm

The Multiplicative Weights algorithm is a special case of Mirror Descent when we choose $\mathcal{G} = \Delta_L$ and choose φ to be the negative Shannon entropy

$$\varphi(g) := \sum_{y=1}^{L} g_y \ln g_y \leq 0 \quad \text{for every } g = (g_1, \ldots, g_L) \in \Delta_L. \tag{25}$$

Its convex conjugate ψ is the following function:

$$\psi(h) = \ln \left(\sum_{y=1}^{L} e^{h_y} \right) \quad \text{for every } h = (h_1, \ldots, h_L) \in \mathbb{R}^L. \tag{26}$$

The gradient of ψ is the softmax function:

$$\nabla\psi(h) = \mathsf{softmax}(h) := \frac{1}{\sum_{y=1}^{L} e^{h_y}} (e^{h_1}, \ldots, e^{h_L}) \in \Delta_L.$$

Each e^{h_y} is often interpreted as the (unnormalized) *weight* on label $y \in [L]$, and the softmax function normalizes these weights to a probability distribution over $[L]$. Thus the additive update on h_y in (23) becomes a multiplicative update on the weight e^{h_y}, hence the name multiplicative weights.

Lemma 2.3. *The function ψ in (26) is 1-smooth in $\|\cdot\|_\infty$. Moreover, the softmax function is a 1-Lipschitz multi-variate function from $\|\cdot\|_\infty$ to $\|\cdot\|_1$. That is, for every $h, \hat{h} \in \mathbb{R}^L$,*

$$\|\mathsf{softmax}(h) - \mathsf{softmax}(\hat{h})\|_1 \leq \|h - \hat{h}\|_\infty.$$

We give a proof of Lemma 2.3 using Pinsker's inequality in Appendix A.1.

3 Regularity Lemma with Weight-Restricted Calibration

As we discuss in Section 1.5, the key to proving our main theorems (Theorems 1.10 and 1.11) is to establish an enhanced regularity lemma that allows us to construct a low complexity function $s : \{0,1\}^n \to \Delta_L$ satisfying both multiaccuracy (12) and weight-restricted calibration (11). We formally state and prove this result in both the non-uniform and uniform settings.

3.1 Non-uniform Setting

Theorem 3.1 (Non-Uniform Regularity Lemma with Weight-Restricted Calibration)**.** *Let $\mathcal{F}$ be a family of functions $f : \{0,1\}^n \to [-1,1]^L$ with circuit complexity at most $T_{\mathcal{F}}$, and let $\mathcal{R}$ be a family of functions $r : \Delta_L \to [-1,1]^L$ with circuit complexity at most $T_{\mathcal{F}}$. Define $T := T_{\mathcal{F}} + T_{\mathcal{R}}$. For every distribution μ on $\{0,1\}^n$, every $g^* : \{0,1\}^n \to \Delta_L$, and every $\varepsilon \in (0, 1/2)$, there exists a simulator $s : \{0,1\}^n \to \Delta_L$ such that*

1. *s has circuit complexity*
$$O\left(\frac{T \log L}{\varepsilon^2} + L \cdot \mathsf{poly}(1/\varepsilon, \log L)\right);$$
2. *s is indistinguishable from g^* w.r.t. $\mathcal{F}$:*
$$|\langle s - g^*, f\rangle| \le \varepsilon \quad \text{for every } f \in \mathcal{F};$$
3. *s satisfies ε-weight-restricted calibration w.r.t. $\mathcal{R}$:*
$$\langle s - g^*, r \circ s\rangle \le \varepsilon \quad \text{for every } r \in \mathcal{R}.$$

We prove Theorem 3.1 by constructing s using Algorithm 1 based on the Multiplicative Weights technique discussed in Section 2.3. In particular, our analysis tracks the Fenchel-Young divergence $\Gamma = \Gamma_{\varphi,\psi}$ defined by the negative Shannon entropy φ in (25) and its convex conjugate ψ in (26). For every $g : \{0,1\}^n \to \Delta_L$ and $h : \{0,1\}^n \to \mathbb{R}$, we define

$$\Gamma(g, h) := \mathbb{E}_{x\sim\mu}[\Gamma(g(x), h(x))].$$

For any norm $\|\cdot\|$ on $\mathbb{R}^L$ and any function $h : \{0,1\}^n \to \mathbb{R}^L$, we define

$$\|h\| := \max_{x\in\{0,1\}^n} \|h(x)\|.$$

As we will show, Theorem 3.1 follows from Lemmas 3.2 and 3.3 below.

Lemma 3.2. *Algorithm 1 terminates after $k = O((\log L)/\varepsilon^2)$ updates. Moreover, the output s produced by the algorithm satisfies*

$$\begin{aligned} |\langle s - g^*, f\rangle| &\le \varepsilon \quad \text{for every } f \in \mathcal{F}, \\ \langle s - g^*, r \circ s\rangle &\le \varepsilon \quad \text{for every } r \in \mathcal{R}. \end{aligned} \tag{27}$$

Proof. As we will show, throughout the algorithm, for every k, we have

$$\Gamma(g^*, h_k) < \Gamma(g^*, h_0) - k \cdot \varepsilon^2/4. \tag{28}$$

The definition of the Fenchel-Young divergence Γ ensures that $\Gamma(g^*, h_k) \ge 0$. We also have

$$\begin{aligned} \Gamma(g^*, h_0) = \mathbb{E}_{x\sim\mu}\Gamma(g^*(x), h_0(x)) &= \mathbb{E}_{x\sim\mu}[\varphi(g^*(x)) + \psi(h_0(x)) - \langle g^*(x), h_0(x)\rangle] \\ &= \mathbb{E}_{x\sim\mu}[\varphi(g^*(x)) + \psi(h_0(x))] \\ &\le \mathbb{E}_{x\sim\mu}[\psi(h_0(x))] \\ &= \ln L. \qquad \text{(by the definitions of } \varphi \text{ and } \psi \text{ in (25) and (26))} \end{aligned}$$

Algorithm 1: Non-uniform Regularity Lemma with Weight-Restricted Calibration

Given: *A family* $\mathcal{F}$ *of functions* $f : \{0,1\}^n \to [-1,1]^L$ *and a family* $\mathcal{R}$ *of functions* $r : \Delta_L \to [-1,1]^L$*; a function* $g^* : \{0,1\}^n \to \Delta_L$*; parameter* $\varepsilon \in (0,1)$*; distribution* μ *on* $\{0,1\}^n$*;*

Initialize $h_0 : \{0,1\}^n \to \mathbb{R}^L$ as the constant function where $h_0(x) = (0,\ldots,0)$ for every $x \in \{0,1\}^n$;

Compute $\hat{g}_0 : \{0,1\}^n \to \Delta_L$ s.t. $\|\hat{g}_0 - g_0\|_1 \le \varepsilon/10$, where $g_0 := \mathsf{softmax} \circ h_0$;

$k \leftarrow 0$;

`updated` $\leftarrow$ `TRUE`;

while `updated` **do**

  `updated` $\leftarrow$ `FALSE`;

  if *there exists* $f \in \mathcal{F} \cup (-\mathcal{F})$ *such that* $\langle \hat{g}_k - g^*, f\rangle > \varepsilon$ **then**

    $h_{k+1} \leftarrow h_k - \varepsilon f$;

    Compute $\hat{g}_{k+1} : \{0,1\}^n \to \Delta_L$ s.t. $\|\hat{g}_{k+1} - g_{k+1}\|_1 \le \varepsilon/10$, where $g_{k+1} := \mathsf{softmax} \circ h_{k+1}$;

    $k \leftarrow k+1$;

    `updated` $\leftarrow$ `TRUE`;

  if *there exists* $r \in \mathcal{R}$ *such that* $\langle \hat{g}_k - g^*, r \circ \hat{g}_k\rangle > \varepsilon$ **then**

    $h_{k+1} \leftarrow h_k - \varepsilon\, r \circ \hat{g}_k$;

    Compute $\hat{g}_{k+1} : \{0,1\}^n \to \Delta_L$ s.t. $\|\hat{g}_{k+1} - g_{k+1}\|_1 \le \varepsilon/10$, where $g_{k+1} := \mathsf{softmax} \circ h_{k+1}$;

    $k \leftarrow k+1$;

    `updated` $\leftarrow$ `TRUE`;

return $s := \hat{g}_k$;

Therefore, (28) implies that $k = O((\log L)/\varepsilon^2)$. Moreover, when Algorithm 1 terminates, both conditions at Lines 8 and 13 are violated, implying (27).

It remains to prove (28). We will prove the stronger claim that whenever we update from h_k to h_{k+1} at Line 9 or Line 14,

$$\Gamma(g^*, h_{k+1}) < \Gamma(g^*, h_k) - \varepsilon^2/4. \tag{29}$$

We focus on the case where the update occurs at Line 9, and the other case of Line 14 can be handled similarly. By Lemma 2.2, Lemma 2.3 and the fact that $\|f\|_\infty \le 1$, we have

$$\langle g_k - g^*, \varepsilon f\rangle \le \Gamma(g^*, h_k) - \Gamma(g^*, h_{k+1}) + \frac{1}{2}\cdot\varepsilon^2. \tag{30}$$

By the condition at Line 8 and the fact that $\|\hat{g}_k - g_k\|_1 \le \varepsilon/10$, we have

$$\langle g_k - g^*, f\rangle > (9/10)\varepsilon. \tag{31}$$

Plugging (31) into (30) proves (29). □

Lemma 3.3 (Controlling circuit complexity)**.** *There exists an implementation of Algorithm 1 such that the following holds. Assume that the classes $\mathcal{F}$ and $\mathcal{R}$ both consist of functions with circuit complexity at most T. Then for every k, the circuit complexity of $\hat{g}_k$ is at most $(k+1)(T + L\cdot \mathsf{poly}(1/\varepsilon, \log L))$, and each coordinate of $\hat{g}_k(x)$ for every $x \in \{0,1\}^n$ is represented in binary format with $O(\log(L/\varepsilon))$ bits.*

We will use the following lemma to prove Lemma 3.3:

Lemma 3.4 (Approximate computation of softmax)**.** *For $\varepsilon \in (0, 1/2), B \ge 2$, let $q, \hat{q} \in [-B, B]^L$ be vectors such that $\|\hat{q} - q\|_\infty \le \varepsilon/3$. Assume that each coordinate of $\hat{q}$ is rational and is represented in binary format with finitely many bits. There exists a circuit of size $L\cdot\mathsf{poly}(B, \log(1/\varepsilon), \log L)$ that, given $\hat{q}$, computes $v \in \Delta_L$ such that $\|v - \mathsf{softmax}(q)\| \le \varepsilon$, where each coordinate of v is represented in binary format with $O(\log(L/\varepsilon))$ bits.*

Proof. Since $\hat{q} \in [-B, B]^L$, the integer part of each coordinate of $\hat{q}$ has $O(\log B)$ bits. We keep only the first $O(\log(1/\varepsilon))$ bits in the fractional part of each coordinate of $\hat{q}$. We can make sure that this truncated $\hat{q}$ still satisfies $\|\hat{q} - q\|_\infty \le \varepsilon/2$. Suppose $\hat{q} = (\hat{q}_1, \ldots, \hat{q}_L)$. For each $y = 1, \ldots, L$, using Taylor's expansion of e^x, we can compute p_y in $\mathsf{poly}(B, \log(1/\varepsilon))$ circuit complexity such that

$$|p_y - e^{\hat{q}_y}|/e^{\hat{q}_y} \in [1 - \varepsilon/10, 1 + \varepsilon/10].$$

This implies that $p_y = e^{\tilde{q}_y}$ for some $\tilde{q}_y \in [\hat{q}_y - \varepsilon/4, \hat{q}_y + \varepsilon/4] \subseteq [q_y - 3\varepsilon/4, q_y + 3\varepsilon/4]$. That is,

$$\|\tilde{q} - q\|_\infty \le 3\varepsilon/4. \tag{32}$$

Define

$$v := \frac{1}{\sum_{y\in[L]} p_y}(p_1, \ldots, p_L) = \mathsf{softmax}(\tilde{q}).$$

Since each p_y can be computed in circuit complexity $\mathsf{poly}(B, \log(1/\varepsilon))$, we have that v can be computed in circuit complexity $L\cdot\mathsf{poly}(B, \log(1/\varepsilon), \log L)$. By (32) and the Lipschitzness of $\mathsf{softmax}$ (Lemma 2.3), we have

$$\|v - \mathsf{softmax}(q)\|_1 = \|\mathsf{softmax}(\tilde{q}) - \mathsf{softmax}(q)\|_1 \le 3\varepsilon/4.$$

The proof is completed by noting that we can round each coordinate of v to the first $O(\log(L/\varepsilon))$ bits while ensuring that the updated v satisfies $v \in \Delta_L$ and $\|v - \mathsf{softmax}(q)\|_1 \le \varepsilon$. □

We are now ready to prove Lemma 3.3.

Proof of Lemma 3.3. Let $\kappa = O((\log L)/\varepsilon^2)$ be an upper bound on the total number of updates in Algorithm 1, as guaranteed by Lemma 3.2. Throughout the algorithm, we maintain a circuit C_k that computes $x \mapsto (\hat{h}_k(x), \hat{g}_k(x))$ where $\hat{h}_k : \{0,1\}^n \to \mathbb{R}^L, \hat{g}_k : \{0,1\}^n \to \Delta_L$ satisfy the following:

1. $\|\hat{h}_k\|_\infty \le k\varepsilon \le \kappa\varepsilon$, so the integer part of each coordinate of $\hat{h}_k$ has at most $O(\log(\kappa\varepsilon))$ bits;
2. $\|\hat{h}_k - h_k\|_\infty \le k \cdot (\varepsilon/30\kappa) \le \varepsilon/30$;
3. $\|\hat{g}_k - g_k\|_\infty \le \varepsilon/10$ as required by Lines 3, 10 and 15;
4. Each coordinate of $\hat{h}_t(x)$ for every $x \in \{0,1\}^n$ is represented in binary format with $O(\log(\kappa/\varepsilon))$ bits after the decimal point;
5. Each coordinate of $\hat{g}_t(x)$ for every $x \in \{0,1\}^n$ is represented in binary format with $O(\log(L/\varepsilon))$ bits.

We show that the circuit complexity of C_k need not exceed $(k+1)(T + L \cdot \mathsf{poly}(1/\varepsilon, \log L))$. It suffices to inductively prove a circuit complexity upper bound T_k for C_k such that

$$\begin{aligned} T_0 &= L \cdot \mathsf{poly}(1/\varepsilon, \log L), \\ T_{k+1} &\le T_k + T + L \cdot \mathsf{poly}(1/\varepsilon, \log L). \end{aligned}$$

We prove this by induction on t. When $t = 0$, the algorithm initializes h_0 to be the constant zero function, so $\hat{h}_0$ can be computed in circuit complexity $O(L)$. By Lemma 3.4, $\hat{g}_0$ can be computed in circuit complexity $L \cdot \mathsf{poly}(\log(1/\varepsilon), \log L)$.

Now suppose we have a circuit C_k with size at most T_k that computes $x \mapsto (\hat{h}_k(x), \hat{g}_k(x))$ satisfying the five conditions above, and let us construct the circuit C_{k+1} that computes $x \mapsto (\hat{h}_{k+1}(x), \hat{g}_{k+1}(x))$. In Algorithm 1, h_{k+1} may be updated from h_k at either Line 9 (using f) or Line 14 (using $r \circ \hat{g}_t$). Both cases can be handled similarly using our assumption that both f and r have circuit complexity at most T. We focus on the first case for simplicity. We first compute $\hat{f}$ with $\|\hat{f}\|_\infty \le \varepsilon$ such that $\|\hat{f} - \varepsilon f\|_\infty \le \varepsilon/30\kappa$, where each coordinate of $\hat{f}(x)$ need only have $O(\log(\kappa/\varepsilon))$ bits in binary representation. This can be done in circuit complexity $T + L \cdot \mathsf{poly}(\log(\kappa/\varepsilon))$, noting that we only need to read $O(\log(\kappa/\varepsilon))$ bits from each coordinate of $f(x)$. Then we subtract $\hat{f}$ from $\hat{h}_k$ to obtain $\hat{h}_{k+1}$. The subtraction takes circuit complexity $L \cdot \mathsf{poly}(\log(\kappa/\varepsilon))$ and ensures that h_{k+1} satisfies conditions 1,2, and 4. Finally, by Lemma 3.4, we can compute $\hat{g}_{k+1}$ satisfying conditions 3 and 5 from $\hat{f}_{k+1}$ in circuit complexity $O(L \cdot \mathsf{poly}(\kappa\varepsilon, \log(1/\varepsilon), \log L))$. The total circuit complexity of C_{k+1} is

$$\begin{aligned} &T_k + T + L \cdot \mathsf{poly}(\log(\kappa/\varepsilon)) + O(L \cdot \mathsf{poly}(\kappa\varepsilon, \log(1/\varepsilon), \log L)) \\ \le\ &T_k + T + L \cdot \mathsf{poly}(1/\varepsilon, \log L), \end{aligned}$$

where we used the fact that $\kappa = O((\log L)/\varepsilon^2)$. □

We now complete the proof of Theorem 3.1 using Lemma 3.2 and Lemma 3.3.

Proof of Theorem 3.1. Lemma 3.2 ensures that the output function s satisfies Conditions 2 and 3 in Theorem 3.1 (indistinguishability and weight-restricted calibration). By Lemma 3.3, we can make sure that s has circuit complexity at most

$$(k+1)(T + L \cdot \mathsf{poly}(1/\varepsilon, \log L)) = O\left(\frac{T \log L}{\varepsilon^2} + L \cdot \mathsf{poly}(1/\varepsilon, \log L)\right),$$

where we used the fact that $k + 1 = O((\log L)/\varepsilon^2)$ by Lemma 3.2. This establishes Condition 1 required by Theorem 3.1. □

3.2 Uniform Setting

To formally state and prove our enhanced regularity lemma in the uniform setting, we need to extend the notions of multiaccuracy (i.e. indistinguishability) and calibration to the uniform setting. To that end, we make the following definitions.

Definition 3.5 (Induced Distribution)**.** *Let μ be a distribution over $\{0,1\}^n$. For a function $g : \{0,1\}^n \to \Delta_L$, we define μ_g as the distribution of $(x, y) \in \{0,1\}^n \times [L]$ where we first draw x from μ and then draw y from the distribution $g(x)$. For a pair of functions $g, g^* : \{0,1\}^n \to \Delta_L$, we define μ_{g,g^*} as the distribution of $(x, y, y^*) \in \{0,1\}^n \times [L] \times [L]$, where we first draw x from μ, and then independently draw $y \sim g(x), y^* \sim g^*(x)$.*

Definition 3.6 (Distinguishing Oracle)**.** *An (m, T)-*distinguishing oracle *is a time-T algorithm $\mathcal{A}$ that takes m data points $(x_1, y_1, y_1^*), \ldots, (x_m, y_m, y_m^*) \in \{0,1\}^n \times [L] \times [L]$ as input, and outputs (a succinct description of) a function $f : \{0,1\}^n \to [-1,1]^L$ such that given any $x \in \{0,1\}^n$, the function value $f(x) \in [-1,1]^L$ can be evaluated in time T.*

Definition 3.7 (Calibration Oracle)**.** *An (m, T)-*calibration oracle *is a time-T algorithm $\mathcal{B}$ that takes m data points $(v_1, y_1, y_1^*), \ldots, (v_m, y_m, y_m^*) \in \Delta_L \times [L] \times [L]$ as input, and outputs (a succinct description of) a function $r : \Delta_L \to [-1,1]^L$ such that given any $v \in \Delta_L$, the function value $r(v) \in [-1,1]^L$ can be evaluated in time T.*

Definition 3.8 (Distinguishability)**.** *Let μ be a distribution over $\{0,1\}^n$ and let $\mathcal{A}$ be an (m, T)-distinguishing oracle. For a pair of functions $g, g^* : \{0,1\}^n \to \Delta_L$, we say g is $(\mathcal{A}, \varepsilon, \delta)$-*distinguishable *from g^* if with probability at least $1-\delta$ over the random draw of m i.i.d. data points $(x_1, y_1, y_1^*), \ldots, (x_m, y_m, y_m^*) \sim \mu_{g,g^*}$ taken as input by oracle $\mathcal{A}$, the output function $f : \{0,1\}^n \to [-1,1]^L$ satisfies*

$$\langle g - g^*, f \rangle > \varepsilon.$$

*Correspondingly, we say g is $(\mathcal{A}, \varepsilon, \delta)$-*indistinguishable *from g^* if it is not $(\mathcal{A}, \varepsilon, \delta)$-distinguishable.*

Definition 3.9 (Calibration)**.** *Let μ be a distribution over $\{0,1\}^n$ and let $\mathcal{B}$ be an (m, T) calibration oracle. For a pair of functions $g, g^* : \{0,1\}^n \to \Delta_L$, we say g is $(\mathcal{B}, \varepsilon, \delta)$-*miscalibrated *w.r.t. g^* if with probability at least $1-\delta$ over the random draw of m i.i.d. data points $(x_1, y_1, y_1^*), \ldots, (x_m, y_m, y_m^*)$ from μ_{g,g^*}, given $(g(x_1), y_1, y_1^*), \ldots, (g(x_m), y_m, y_m^*)$ as input to $\mathcal{B}$, the output function $r : \Delta_L \to [-1,1]^L$ satisfies*

$$\langle g - g^*, r \circ g \rangle > \varepsilon.$$

*Correspondingly, we say g is $(\mathcal{B}, \varepsilon, \delta)$-*calibrated *w.r.t. g^* if it is not $(\mathcal{B}, \varepsilon, \delta)$-miscalibrated.*

In Definitions 3.8 and 3.9 above, the success probability of an oracle can be easily amplified by independent repetition. Suppose we have an oracle with success probability $\alpha > 0$. We can construct an oracle with success probability at least $1 - \delta$ by independently running the oracle $u = O(\alpha^{-1} \log(1/\delta))$ times to ensure that with probability at least $1 - \delta/2$, at least one of the runs is successful. We can then perform a simple test using $O(\varepsilon^{-2} \log(u/\delta)))$ additional fresh data points to select a successful run with probability at least $1 - \delta/2$ (with a constant-factor loss in ε). The overall success probability is at least $1 - \delta$ by the union bound.

We are now ready to state the enhanced regularity lemma in the uniform setting:

Theorem 3.10 (Uniform Regularity Lemma with Weight-Restricted Calibration)**.** *Let $\mathcal{A}$ be an (m, T) distinguishing oracle and let $\mathcal{B}$ be an (m, T) calibration oracle. Then there exists a simulating algorithm $\mathcal{S}$ using $\mathcal{A}$ and $\mathcal{B}$ as subroutines such that for every $g^* : \{0,1\}^n \to \Delta_L$ the following holds:*

1. *Given $\varepsilon, \delta \in (0, 1/2)$ as input, $\mathcal{S}$ addtionally takes $\mathsf{poly}(m, \log L, 1/\varepsilon, \log(1/\delta))$ i.i.d. data points from μ_{g^*} as input, and returns a function $s : \{0,1\}^n \to \Delta_L$ in time $\mathsf{poly}(m, T, L, 1/\varepsilon, \log(1/\delta))$. Moreover, given an arbitrary $x \in \{0,1\}^n$, the function value $s(x)$ can be evaluated also in time $\mathsf{poly}(m, T, L, 1/\varepsilon, \log(1/\delta))$.*

2. *With probability at least $1 - \delta$ over the random draw of the data points, the output function s is simultaneously $(\mathcal{A}, \varepsilon, \Omega(\delta\varepsilon^2/\log L))$-indistinguishable from g^* and $(\mathcal{B}, \varepsilon, \Omega(\delta\varepsilon^2/\log L))$-calibrated w.r.t. g^*.*

We prove Theorem 3.10 using Algorithm 2 as the simulating algorithm $\mathcal{S}$ based on the same multiplicative weights idea as Algorithm 1. The difference is that instead of directly looking for f and r as in Lines 8 and 13 of Algorithm 1, we obtain them efficiently by invoking the oracles $\mathcal{A}$ and $\mathcal{B}$ at Lines 10 and 19 in Algorithm 2. Recall that for $y \in [L]$, we use $\mathbf{e}_y$ to denote the standard basis vector whose y-th coordinate is one. This ensures that for every $v \in \Delta_L$, we have $\mathbb{E}_{y \sim v}[\mathbf{e}_y] = v$.

Theorem 3.10 follows immediately from the following lemma about Algorithm 2:

Lemma 3.11. *For every $\varepsilon \in (0, 1/2)$, there exist $\kappa = O((\log L)/\varepsilon^2), \delta' = \Omega(\delta/\kappa), m' = O((\log(1/\delta'))/\varepsilon^2)$ such that when we run Algorithm 2 with parameters (ε, m'), with probability at least $1-\delta$, the following holds. Algorithm 2 terminates after $k \le \kappa$ updates and returns a function $s : \{0,1\}^n \to [-1,1]^L$ that is simultaneously $(\mathcal{A}, \varepsilon, \delta')$-indistinguishable from g^* and $(\mathcal{B}, \varepsilon, \delta')$-calibrated w.r.t. g^*. Also, Algorithm 2 runs in time $\mathsf{poly}(m, T, L, 1/\varepsilon, \log(1/\delta))$, and for every $x \in \{0,1\}^n$, the function value $s(x)$ can be evaluated in time $\mathsf{poly}(m, T, L, 1/\varepsilon, \log(1/\delta))$.*

Proof. By the Chernoff bound, choosing $m' = O((\log(1/\delta'))/\varepsilon^2)$ is sufficient to ensure that each time Line 11 is performed, with probability at least $1 - \delta'$ we have

$$\left| \frac{1}{m'} \sum_{i=1}^{m'} \langle \hat{g}_k(x_i) - \mathbf{e}_{y_i^*}, f(x_i) \rangle - \langle \hat{g}_k - g^*, f \rangle \right| < \varepsilon/4. \tag{33}$$

Similarly, each time when Line 20 is performed, with probability at least $1 - \delta'$ we have

$$\left| \frac{1}{m'} \sum_{i=1}^{m'} \langle \hat{g}_k(x_i) - \mathbf{e}_{y_i^*}, r(\hat{g}_k(x_i)) \rangle - \langle \hat{g}_k - g^*, r \circ \hat{g}_k \rangle \right| < \varepsilon/4. \tag{34}$$

Algorithm 2: Uniform Regularity Lemma with Weight-Restricted Calibration

Given: *An (m,T)-distinguishing oracle $\mathcal{A}$ and an (m,T)-calibration oracle $\mathcal{B}$ on domain $\{0,1\}^n$ and label space $[L]$; access to i.i.d. data points drawn from μ_{g^*} for distribution μ on $\{0,1\}^n$ and function $g^* : \{0,1\}^n \to \Delta_L$; parameters $\varepsilon \in (0,1), m' \in \mathbb{Z}_{>0}$;*

Initialize $h_0 : \{0,1\}^n \to \mathbb{R}^L$ as the constant function where $h_0(x) = (0,\ldots,0)$ for every $x \in \{0,1\}^n$;

Compute $\hat{g}_0 : \{0,1\}^n \to \Delta_L$ s.t. $\|\hat{g}_0 - g_0\|_1 \le \varepsilon/20$, where $g_0 := \mathsf{softmax} \circ h_0$;

$k \leftarrow 0$;

`updated` $\leftarrow$ `TRUE`;

while `updated` **do**

  `updated` $\leftarrow$ `FALSE`;

  Draw fresh i.i.d. data points $(x_1, y_1^*), \ldots, (x_m, y_m^*)$ from μ_{g^*};

  Draw $y_i \sim \hat{g}_k(x_i)$ independently for $i = 1, \ldots, m$;

  Invoke distinguishing oracle $\mathcal{A}$ on data points (x_i, y_i, y_i^*) and obtain its output $f : \{0,1\}^n \to [-1,1]^L$;

  Draw fresh i.i.d. data points $(x_1, y_1^*), \ldots, (x_{m'}, y_{m'}^*)$ from μ_{g^*} ;

  if $\frac{1}{m'} \sum_{i=1}^{m'} \langle \hat{g}_k(x_i) - \mathbf{e}_{y_i^*}, f(x_i) \rangle > 3\varepsilon/4$ **then**

    $h_{k+1} \leftarrow h_k - (\varepsilon/2) f$;

    Compute $\hat{g}_{k+1} : \{0,1\}^n \to \Delta_L$ s.t. $\|\hat{g}_{k+1} - g_{k+1}\|_1 \le \varepsilon/20$, where $g_{k+1} := \mathsf{softmax} \circ h_{k+1}$;

    $k \leftarrow k + 1$;

    `updated` $\leftarrow$ `TRUE`;

  Draw fresh i.i.d. data points $(x_1, y_1^*), \ldots, (x_m, y_m^*)$ from μ_{g^*} ;

  Draw $y_i \sim \hat{g}_t(x_i)$ independently for $i = 1, \ldots, m$;

  Invoke calibration oracle $\mathcal{B}$ on data points $(\hat{g}_k(x_i), y_i, y_i^*)$ and obtain its output $r : \Delta_L \to [-1,1]^L$;

  Draw fresh i.i.d. data points $(x_1, y_1^*), \ldots, (x_{m'}, y_{m'}^*)$ from μ_{g^*} ;

  if $\frac{1}{m'} \sum_{i=1}^{m'} \langle \hat{g}_k(x_i) - \mathbf{e}_{y_i^*}, r(\hat{g}_k(x_i)) \rangle > 3\varepsilon/4$ **then**

    $h_{k+1} \leftarrow h_k - (\varepsilon/2)\, r \circ \hat{g}_k$;

    Compute $\hat{g}_{k+1} : \{0,1\}^n \to \Delta_L$ s.t. $\|\hat{g}_{k+1} - g_{k+1}\|_1 \le \varepsilon/20$, where $g_{k+1} := \mathsf{softmax} \circ h_{k+1}$;

    $k \leftarrow k + 1$;

    `updated` $\leftarrow$ `TRUE`;

return $s := \hat{g}_k$;

By the definition of $(\mathcal{A}, \varepsilon, \delta')$-distinguishability (Definition 1.6), each time when Line 10 is performed with $\hat{g}_k$ being $(\mathcal{A}, \varepsilon, \delta')$-distinguishable, with probability at least $1 - \delta'$ we have

$$\langle \hat{g}_k - g^*, f \rangle > \varepsilon. \tag{35}$$

Similarly, by the definition of $(\mathcal{B}, \varepsilon, \delta')$-miscalibration Definition 3.9, each time when Line 19 is performed with $\hat{g}_k$ being $(\mathcal{B}, \varepsilon, \delta')$-miscalibrated, with probability at least $1 - \delta'$ we have

$$\langle \hat{g}_k - g^*, r \circ \hat{g}_k \rangle > \varepsilon. \tag{36}$$

By the union bound, we can choose $\delta' = \Omega(\delta/\kappa)$ to ensure that with probability at least $1 - \delta$, all of the above are satisfied *simultaneously* in the first $\kappa + 1$ iterations of the while loop. It thus remains to show that when this occurs, the following two conditions required by Lemma 3.11 are both satisfied:

1. Algorithm 2 terminates after $k \leq \kappa = O((\log L)/\varepsilon^2)$ updates and returns a function $s : \{0,1\}^n \to [-1,1]^L$ that is simultaneously $(\mathcal{A}, \varepsilon, \delta')$-indistinguishable from g^* and $(\mathcal{B}, \varepsilon, \delta')$-calibrated w.r.t. g^*.

2. Algorithm 2 runs in time $\mathsf{poly}(m, T, L, 1/\varepsilon, \log(1/\delta))$, and for every $x \in \{0,1\}^n$, the function value $s(x)$ can be evaluated in time $\mathsf{poly}(m, T, L, 1/\varepsilon, \log(1/\delta))$.

To prove that the output function $s = \hat{g}_k$ is $(\mathcal{A}, \varepsilon, \delta')$-indistinguishable from g^* and $(\mathcal{B}, \varepsilon, \delta')$-calibrated as require by Item 1, it suffices to show that (35) and (36) are both violated. By (33) and (34), it suffices to show that the conditions in the "if" clauses at Lines 12 and 21 are both violated. This is indeed true for the final output $s = \hat{g}_k$ because otherwise Algorithm 2 would proceed to the next iteration of the "while" loop. Similarly, by (33), if the condition at Line 12 is satisfied, we have

$$\langle \hat{g}_k - g^*, f \rangle > \varepsilon/2.$$

By (34), if the condition at Line 21 is satisfied, we have

$$\langle \hat{g}_k - g^*, r \circ \hat{g}_k \rangle > \varepsilon/2.$$

This allows us to prove that Algorithm 2 terminates after $k \leq \kappa = O((\log L)/\varepsilon^2)$ updates following the same idea in the proof of Lemma 3.2, completing the proof of Item 1. Also, Item 2 can be proved similarly to Lemma 3.3 using the fact that both $\mathcal{A}$ and $\mathcal{B}$ run in time T and their output functions f and r can be evaluated also in time T, according to Definitions 3.6 and 3.7. We omit the details. □

4 Generalized and Unified Pseudoentropy Characterizations

In this section, we complete the proofs of our main results Theorems 1.10 and 1.11 using the enhanced regularity lemmas Theorems 3.1 and 3.10 from the previous section.

4.1 Non-uniform Setting

Proof of Theorem 1.10. For $\varphi \in \Phi$, by our assumption that every $\nabla\varphi$ can be efficiently approximated within ℓ_∞ error $\varepsilon/4$, there exists a function $r_\varphi : \Delta_L \to [-1,1]^L$ computable in nonuniform time T_Φ such that

$$\|r_\varphi(v) - \nabla\varphi(v)\|_\infty \le \varepsilon/4 \quad \text{for every } v \in \Delta_L. \tag{37}$$

By (9), the guarantee (4) is equivalent to

$$\langle s - g^*, \nabla\varphi \circ s\rangle \le \varepsilon \quad \text{for every } \varphi \in \Phi.$$

By (37), a sufficient condition for the guarantee (4) is

$$\langle s - g^*, r_\varphi \circ s\rangle \le \varepsilon/2 \quad \text{for every } \varphi \in \Phi.$$

Now the first half of Theorem 1.10 (up to Equation (4)) follows directly from Theorem 3.1, where we set $\mathcal{R} := \{r_\varphi\}_{\varphi\in\Phi}, T_\mathcal{R} := T_\Phi$, set $\mathcal{F}$ to be the family of functions $f : \{0,1\}^n \to [-1,1]^L$ with circuit complexity at most T, set $T_\mathcal{F} := T$, and replace ε with $\varepsilon/2$.

To prove the second half of Theorem 1.10 (i.e. the reverse direction), we note the following identity for every $g, g^*, s : \{0,1\}^n \to \Delta_L$:

$$\mathsf{D}_\varphi(g^*\|g) = \mathsf{D}_\varphi(s\|g) + \langle s - g^*, \nabla\varphi \circ g\rangle + (\mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*)).$$

This identity follows immediately from the definitions of H_φ and D_φ in Definitions 1.7 and 1.8. By (37) and the fact that $\mathsf{D}_\varphi(s\|g) \ge 0$, we have

$$\begin{aligned}\mathsf{D}_\varphi(g^*\|g) &\ge \langle s - g^*, \nabla\varphi \circ g\rangle + (H_\varphi(s) - H_\varphi(g^*)) \\ &\ge \langle s - g^*, r_\varphi \circ g\rangle + (H_\varphi(s) - H_\varphi(g^*)) - \varepsilon/2. \end{aligned}\tag{38}$$

Since r_φ is computable in non-uniform time T_Φ, when g is computable in non-uniform time T, we have that $r_\varphi \circ g$ is computable in non-uniform time $O(T + T_\Phi)$, so for $s \in \mathsf{Ind}(g^*, O(T + T_\Phi), \varepsilon/2)$, we have

$$\langle s - g^*, r_\varphi \circ g\rangle \ge -\varepsilon/2.$$

Plugging this into (38) proves (5). □

4.2 Uniform Setting

We need the following definition to give a formal statement of Theorem 1.11.

Definition 4.1 (Weak Agnostic Calibration Oracle)**.** *Let μ be a fixed distribution on $\{0,1\}^n$. Let $\mathcal{R}$ be a family of functions $r : \Delta_L \to [-1,1]^L$, and let $\varepsilon_1, \varepsilon, \delta \in (0,1)$ be parameters. We say an (m,T)-calibration oracle $\mathcal{B}$ is an $(\mathcal{R}, \varepsilon_1, \varepsilon, \delta)$-*weak agnostic calibration oracle *if the following holds. For every pair $g, g^* : \{0,1\}^n \to \Delta_L$, if there exists $r \in \mathcal{R}$ such that*

$$\langle g - g^*, r \circ g\rangle > \varepsilon_1,$$

then with probability at least $1 - \delta$ over the random draw of m i.i.d. data points $(x_1, y_1, y_1^*), \ldots, (x_m, y_m, y_m^*) \sim \mu_{g,g^*}$, given $(g(x_1), y_1, y_1^*), \ldots, (g(x_m), y_m, y_m^*)$ as input to $\mathcal{B}$, the output function $r_{\mathsf{output}} : \Delta_L \to [-1,1]^L$ satisfies

$$\langle g - g^*, r_{\mathsf{output}} \circ g\rangle > \varepsilon.$$

The above definition is analogous to the standard definition of weak agnostic learning [KMV08, Fel10]. When $\mathcal{R}$ is a finite class, we can implement an $(\mathcal{R}, \varepsilon_1, \varepsilon_1/2, \delta)$-weak agnostic calibration oracle using $m = O(\varepsilon_1^{-2} \log(|\mathcal{R}|/\delta))$ data points by performing *empirical risk minimization (ERM)*, where we output $r \in \mathcal{R}$ that maximizes the empirical correlation:

$$\frac{1}{m} \sum_{i=1}^{m} \langle g(x_i) - \mathbf{e}_{y_i^*}, r(g(x_i)) \rangle.$$

The following claim follows immediately from Definitions 3.9 and 4.1:

Claim 4.2. *Let $\mathcal{R}$ be a family of functions $r : \Delta_L \to \mathbb{R}^L$ and $\mathcal{B}$ be an $(\mathcal{R}, \varepsilon_1, \varepsilon, \delta)$-weak agnostic calibration oracle. For functions $g, g^* : \{0,1\}^n \to \Delta_L$, if g is $(\mathcal{B}, \varepsilon, \delta)$-calibrated w.r.t. g^*, then*

$$\langle g - g^*, r \circ g \rangle \le \varepsilon_1 \quad \textit{for every } r \in \mathcal{R}.$$

Theorem 4.3 (Formal statement of Theorem 1.11)**.** *Consider domain $\{0,1\}^n$ and label space $[L]$. Let Φ be a family of convex functions $\varphi : \Delta_L \to \mathbb{R}$. For every $\varepsilon, \delta \in (0, 1/2)$, there exists $\delta' = \Omega(\delta\varepsilon^2/\log L)$ and a simulating algorithm $\mathcal{S}$ such that the following holds for every $g^* : \{0,1\}^n \to \Delta_L$ and oracles $\mathcal{A}$ and $\mathcal{B}$:*

1. *If $\mathcal{A}$ is an (m, T) distinguishing oracle and $\mathcal{B}$ is an (m, T) calibration oracle, then $\mathcal{S}$ takes $\mathsf{poly}(m, \log L, 1/\varepsilon, \log(1/\delta))$ i.i.d. data points from μ_{g^*} as input, uses $\mathcal{A}$ and $\mathcal{B}$ as oracles, and returns a function $s : \{0,1\}^n \to \Delta_L$ in time $\mathsf{poly}(m, T, L, 1/\varepsilon, \log(1/\delta))$. Moreover, given an arbitrary $x \in \{0,1\}^n$, the function value $s(x)$ can be evaluated also in time $\mathsf{poly}(m, T, L, 1/\varepsilon, \log(1/\delta))$.*
2. *If in addition $\mathcal{B}$ is an $(\{\nabla\varphi\}_{\varphi\in\Phi}, \varepsilon_1, \varepsilon, \delta')$ weak agnostic calibration oracle for some $\varepsilon_1 > 0$, then with probability at least $1 - \delta$ over the random draw of the input data points, the output function s is $(\mathcal{A}, \varepsilon, \delta')$-indistinguishable from g^* and satisfies*

$$H_\varphi(s) - H_\varphi(g^*) \ge D_\varphi(g^* \| s) - \varepsilon_1 \quad \textit{for every } \varphi \in \Phi. \tag{39}$$

Proof. By (9), the guarantee (39) is equivalent to

$$\langle s - g^*, \nabla\varphi \circ s \rangle \le \varepsilon_1 \quad \text{for every } \varphi \in \Phi.$$

The theorem now follows directly from Theorem 3.10 and Claim 4.2, where we choose $\mathcal{R} = \{\nabla\varphi\}_{\varphi\in\Phi}$. □

5 Handling Unbounded Subgradients

Our Theorem 1.10 requires the assumption that the subgradient $\nabla\varphi$ is bounded. This assumption is not satisfied by the Shannon entropy, where $\varphi(v) = \sum v_i \ln v_i$. In this case, $\nabla\varphi(v) = (\ln(v_1), \ldots, \ln(v_L))$ becomes unbounded when some v_i approaches zero. We show that this can be easily fixed using the following more general theorem than Theorem 1.10. Here, we consider an arbitrary transformation $\sigma_\varphi : \Delta_L \to \Delta_L$ associated with each $\varphi \in \Phi$, and Theorem 1.10 is the special case when these transformations σ_φ are the identity function. We will later choose these transformations so that the subgradient $\nabla\varphi(\sigma_\varphi(v))$ at the transformed point $\sigma_\varphi(v)$ is bounded, even when $\nabla\varphi(v)$ may not be bounded.

Theorem 5.1. *Let $T > 0$ be a time bound and let $\varepsilon \in (0,1)$ be an error parameter. Let Φ be a family of convex functions $\varphi : \Delta_L \to \mathbb{R}$. For every $\varphi \in \Phi$, let $\sigma_\varphi : \Delta_L \to \Delta_L$ be an arbitrary transformation. Assume that given $v \in \Delta_L$, the transformed subgradient $\nabla\varphi(\sigma_\varphi(v))$ is bounded in $[-1,1]^L$ and can be computed to ℓ_∞ accuracy $\varepsilon/4$ in nonuniform time T_Φ, for every $\varphi \in \Phi$. Then there exists*

$$T' = O\left(\frac{(T + T_\Phi)\log L}{\varepsilon^2} + L \cdot \mathsf{poly}(1/\varepsilon, \log L)\right) \tag{40}$$

such that for every $g^* : X \to \Delta_L$, there exists $s \in \mathsf{Ind}(g^*; T, \varepsilon) \cap \mathsf{Time}(T')$ such that

$$\mathsf{H}_\varphi(\sigma_\varphi \circ s) - \mathsf{H}_\varphi(g^*) \geq D_\varphi(g^* \| \sigma_\varphi \circ s) - \varepsilon \quad \textit{for every } \varphi \in \Phi. \tag{41}$$

Conversely, for every $T > 0$, there exists $T' = O(T + T_\Phi)$ such that for every $g^* : X \to \Delta_L$ and every $\varphi \in \Phi$,

$$\max_{s \in \mathsf{Ind}(g^*; T', \varepsilon/2)} (\mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*)) \leq \min_{g \in \mathsf{Time}(T)} \mathsf{D}_\varphi(g^* \| \sigma_\varphi \circ g) + \varepsilon. \tag{42}$$

Theorem 5.1 can be proved in the exact same way as Theorem 1.10 using the enhanced regularity lemma Theorem 3.1, and similarly, we can prove a generalized version of Theorem 4.3 using Theorem 3.10 to handle unbounded subgradients in the uniform setting.

The advantage of using Theorem 5.1 is that it only requires $\nabla\varphi(\sigma_\varphi(v))$ to be bounded after the transformation $v \mapsto \sigma_\varphi(v)$. The challenge, however, is that while (41) is stated for the transformed $\sigma_\varphi \circ s$, Theorem 5.1 only ensures that the un-transformed s belongs to $\mathsf{Ind}(g^*; T, \varepsilon) \cap \mathsf{Time}(T')$. This mis-alignment between s and $\sigma_\varphi \circ s$ also appears in the reverse direction (42) where we minimize the divergence of the transformed $\sigma_\varphi \circ g$ subject to the un-transformed g being in $\mathsf{Time}(T)$. Below we show how to address this mis-alignment challenge for the Shannon entropy by choosing the transformation σ_φ appropriately.

For the (negative) Shannon entropy φ in (25), define $\sigma_\varphi(v) = (1-\varepsilon)v + \varepsilon u$, where $u = (1/L, \ldots, 1/L)$ is the uniform distribution and $\varepsilon \in (0, 1/2)$. Intuitively, this transformation perturbs v away from the boundary of Δ_L. We now have $\|\nabla\varphi(\sigma_\varphi(v))\|_\infty \leq \ln(L/\varepsilon)$, so the boundedness assumption of Theorem 5.1 is satisfied after a multiplicative scaling of $1/\ln(L/\varepsilon)$. Moreover, we have the following implication connecting $\sigma_\varphi \circ s$ with s that addresses the mis-alignment challenge:

$$s \in \mathsf{Ind}(g^*; T, \varepsilon) \cap \mathsf{Time}(T') \implies \sigma_\varphi \circ s \in \mathsf{Ind}(g^*; T, 3\varepsilon) \cap \mathsf{Time}(O(T' \,\mathsf{polylog}(L/\varepsilon))).$$

Indeed, it is clear that $\|\sigma_\varphi(v) - v\|_1 \leq 2\varepsilon$, so $s \in \mathsf{Ind}(g^*; T, \varepsilon)$ implies that $\sigma_\varphi \circ s \in \mathsf{Ind}(g^*; T, 3\varepsilon)$. Also, given that s can be computed in circuit complexity T', it is clear that $\sigma_\varphi \circ s$ can be computed in circuit complexity $O(T' \,\mathsf{polylog}(L/\varepsilon))$. We can similarly address the mis-alignment challenge in the reverse direction (42) using the following connection between $\sigma_\varphi \circ g$ and g:

$$\mathsf{D}_\varphi(g^* \| \sigma_\varphi \circ g) \leq \mathsf{D}_\varphi(g^* \| g) + O(\varepsilon),$$

which holds because for every $v^*, v \in \Delta_L$,

$$\mathsf{D}_\varphi(v^* \| \sigma_\varphi(v)) = \sum_i v_i^* \ln \frac{v_i^*}{\sigma_\varphi(v_i)} \leq \sum_i v_i^* \ln \frac{v_i^*}{(1-\varepsilon)v_i} = \mathsf{D}_\varphi(v^* \| v) + \ln \frac{1}{1-\varepsilon} \leq \mathsf{D}_\varphi(v^* \| v) + O(\varepsilon).$$

6 Exponential Lower Bound for a Particular Entropy Notion

We prove Theorem 1.12 and Corollary 1.13 in this section. We need the following helper lemma:

Lemma 6.1. *[[EFF85]] Define $\alpha = 1/16$. Let n be a sufficiently large positive integer. Then there exist $m = \exp(\Omega(n))$ subsets $S_1, \ldots, S_m \subseteq [n] := \{1, \ldots, n\}$ such that*

1. *$|S_i| \geq \alpha n$ for every $i = 1, \ldots, m$;*
2. *$|S_i \cap S_j| \leq 2\alpha^2 n$ for every distinct $i, j \in \{1, \ldots, m\}$;*
3. *$|S_i|$ is even for every $i = 1, \ldots, m$.*

This lemma can be proved by a natural argument using the probabilistic method (see e.g. Problem 3.2 of [Vad12]).

Proof of Theorem 1.12. Every $x \in \{0,1\}^n = \{0,1\}^L$ can be viewed as the indicator of a subset S_x of $[L]$ where for every $y \in [L]$,

$$y \in S_x \iff x_y = 1.$$

We define $\alpha := 1/16$. By Lemma 6.1, there exists a subset $\mathcal{X}' \subseteq \{0,1\}^n$ such that

1. $|\mathcal{X}'| = \exp(\Omega(n))$;
2. $|S_x| \geq \alpha L$ for every $x \in \mathcal{X}'$;
3. $|S_x \cap S_{x'}| \leq 2\alpha^2 L$ for every distinct $x, x' \in \mathcal{X}'$;
4. $|S_x|$ is even for every $x \in \mathcal{X}'$.

For every $x \in \mathcal{X}'$, we arbitrarily partition S_x into two equal sized subsets $S_x^{(0)}$ and $S_x^{(1)}$. We use $x^{(0)}, x^{(1)} \in \{0,1\}^n$ to denote the indicators of $S_x^{(0)}, S_x^{(1)}$, respectively. That is, $S_{x^{(0)}} = S_x^{(0)}, S_{x^{(1)}} = S_x^{(1)}$.

We choose μ to be the uniform distribution on $\mathcal{X}'$. Let $\eta : \mathcal{X}' \to \{0,1\}$ be a function that we will determine later. For every $x \in \mathcal{X}'$, we define $g^*(x) \in \Delta_L$ to be the uniform distribution on $S_x^{(\eta(x))} \subseteq [L]$. For every $x \in \{0,1\}^n \setminus \mathcal{X}'$, we define $g^*(x)$ arbitrarily. As required in the theorem statement, we define $f : \{0,1\}^n \to [-1,1]^L$ to be the identity function: $f(x) = x \in \{0,1\}^n \subseteq [-1,1]^L$ for every $x \in \{0,1\}^n$. We choose the convex function $\varphi : \Delta_L \to \mathbb{R}$ and its corresponding subgradient as follows: for every $v \in \Delta_L$,

$$\begin{aligned} \varphi(v) &:= \max_{x \in \mathcal{X}'} \langle v, x^{(1-\eta(x))} \rangle, \\ \nabla\varphi(v) &:= x_v^{(1-\eta(x_v))} \in \{0,1\}^L, \quad \text{where } x_v = \arg\max_{x \in \mathcal{X}'} \langle v, x^{(1-\eta(x))} \rangle. \end{aligned}$$

We break ties arbitrarily in the arg max above.

We first show that

$$-\mathsf{H}_\varphi(g^*) \leq 4\alpha. \tag{43}$$

By the definition of H_φ, we have

$$-\mathsf{H}_\varphi(g^*) = \mathbb{E}_{x' \sim \mu} \left[\max_{x \in \mathcal{X}'} \langle g^*(x'), x^{(1-\eta(x))} \rangle \right]. \tag{44}$$

Recall that $g^*(x')$ is the uniform distribution on $S_{x'}^{(\eta(x'))}$, so every coordinate of $g^*(x')$ is at most $1/|S_{x'}^{(\eta(x'))}| \le 1/(\alpha L/2)$, and all coordinates out of $S_{x'}^{(\eta(x'))}$ are zero. Therefore,

$$\langle g^*(x'), x^{(1-\eta(x))}\rangle \le \frac{1}{\alpha L/2} \cdot |S_{x'}^{(\eta(x'))} \cap S_x^{(1-\eta(x))}|. \tag{45}$$

If $x = x'$, we have $|S_{x'}^{(\eta(x'))} \cap S_x^{(1-\eta(x))}| = 0$. If $x \neq x'$, we have

$$|S_{x'}^{(\eta(x'))} \cap S_x^{(1-\eta(x))}| \le |S_{x'} \cap S_x| \le 2\alpha^2 L.$$

Plugging this into (45), we get

$$\langle g^*(x'), x^{(1-\eta(x))}\rangle \le 4\alpha.$$

Combining this with (44), we get (43), as desired.

Now let $s : \{0,1\}^n \to \Delta_L$ be an arbitrary function that is $(\{f\}, 0.05)$ indistinguishable from g^* and satisfies (7). By the indistinguishability, we have

$$\mathbb{E}_{x\sim\mu}\langle s(x), x\rangle = \mathbb{E}_{x\sim\mu}\langle s(x), f(x)\rangle \ge \mathbb{E}_{x\sim\mu}\langle g^*(x), f(x)\rangle - 0.05 = 1 - 0.05.$$

By (7),

$$\mathbb{E}_{x\sim\mu}\langle s(x), x^{(1-\eta(x))}\rangle = -\mathsf{H}_\varphi(s) \le -\mathsf{H}_\varphi(g^*) + 0.05 \le 4\alpha + 0.05 = 1/4 + 0.05.$$

Combining the two inequalities above, we have

$$\mathbb{E}_{x\sim\mu}\langle s(x), x^{(\eta(x))}\rangle = \mathbb{E}_{x\sim\mu}\langle s(x), x\rangle - \mathbb{E}_{x\sim\mu}\langle s(x), x^{(1-\eta(x))}\rangle \ge 0.6. \tag{46}$$

It remains to show that there exists a function $\eta : \mathcal{X}' \to \{0,1\}$ such that every function $s : \{0,1\}^n \to \Delta_L$ satisfying (46) must have circuit complexity $\exp(\Omega(n))$. Let T be a time bound such that for every function $\eta : \mathcal{X}' \to \{0,1\}$, there exists a function $s : \{0,1\}^n \to \Delta_L$ satisfying (46) with circuit complexity at most T (i.e., $s \in \mathsf{Time}(T)$). Our goal is to show that $T = \exp(\Omega(n))$.

We use a probabilistic counting argument. For every fixed function $s : \{0,1\}^n \to \Delta_L$ and every fixed $x \in \mathcal{X}'$, for a uniformly random $\eta : \mathcal{X}' \to \{0,1\}$, we have

$$\mathbb{E}_\eta\langle s(x), x^{(\eta(x))}\rangle = \frac{1}{2}(\langle s(x), x^{(0)}\rangle + \langle s(x), x^{(1)}\rangle) = \frac{1}{2}\langle s(x), x\rangle \le 0.5.$$

Since $\eta(x)$ is distributed independently for different $x \in \mathcal{X}'$, by the Chernoff bound, for every fixed $s : \{0,1\}^n \to \Delta_L$,

$$\Pr_\eta[\mathbb{E}_{x\sim\mu}\langle s(x), x^{(\eta(x))}\rangle \ge 0.6] \le \exp(-\Omega(|\mathcal{X}'|)).$$

There are $\exp(\mathsf{poly}(T))$ circuits of size at most T. Therefore, by the union bound,

$$1 = \Pr_\eta[\exists s \in \mathsf{Time}(T) \text{ s.t. } \mathbb{E}_{x\sim\mu}\langle s(x), x^{(\eta(x))}\rangle \ge 0.6] \le \exp(-\Omega(|\mathcal{X}'|)) \cdot \exp(\mathsf{poly}(T)).$$

Since $|\mathcal{X}'| = \exp(\Omega(n))$, the above inequality implies that $T = \exp(\Omega(n))$ for every sufficiently large n, as desired. □

Proof of Corollary 1.13. It suffices to show that the two conditions (multiaccuracy and weight-restricted calibration) satisfied by s imply that s belongs to $\mathsf{Ind}(g^*; \{f\}, 0.05)$ and satisfies (7) as required by Theorem 1.12.

Indeed, the multiaccuracy condition is equivalent to $s \in \mathsf{Ind}(g^*; \{f\}, 0.05)$. By (9), the weight-restricted calibration condition is equivalent to

$$\mathsf{H}_\varphi(s) - \mathsf{H}_\varphi(g^*) \ge D_\varphi(g^* \| s) - 0.05,$$

which, by the non-negativity of the Bregman divergence, implies (7), as desired. □

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A Deferred Proofs

A.1 Proof of Lemma 2.3

We need the following standard definition of strong convexity and two helper claims (Claims A.2 and A.3).

Definition A.1 (Strong Convexity)**.** *Let $\mathcal{G} \subseteq \mathbb{R}^L$ be a bounded convex set. For $\lambda \geq 0$ and a norm $\|\cdot\|$ on $\mathbb{R}^L$, we say a convex function $\varphi : \mathcal{G} \to \mathbb{R}$ is* λ-strongly convex *w.r.t. $\|\cdot\|$ if for every $g_0, g \in \mathcal{G}$, letting $h_0 \in \mathbb{R}^L$ be an arbitrary subgradient of φ at g_0, we have*

$$\varphi(g) \geq \varphi(g_0) + \langle g - g_0, h_0 \rangle + \frac{\lambda}{2}\|g - g_0\|^2. \tag{47}$$

Claim A.2. *Let $\mathcal{G} \subseteq \mathbb{R}^L$ be a bounded convex set. Let $\varphi : \mathcal{G} \to \mathbb{R}$ be a closed convex function, and let $\psi : \mathbb{R}^L \to \mathbb{R}$ be its convex conjugate. Suppose $(g_0, h_0) \in \mathcal{G} \times \mathbb{R}^L$ is a conjugate pair (i.e. $\Gamma_{\varphi,\psi}(g_0, h_0) = 0$). Then for every $g \in \mathcal{G}$ and $h \in \mathbb{R}^L$,*

$$\Gamma(g, h_0) = \varphi(g) - \varphi(g_0) - \langle g - g_0, h_0 \rangle, \tag{48}$$
$$\Gamma(g_0, h) = \psi(h) - \psi(h_0) - \langle g_0, h - h_0 \rangle. \tag{49}$$

Proof. Since (g_0, h_0) is a conjugate pair, we have

$$0 = \Gamma_{\varphi,\psi}(g_0, h_0) = \varphi(g_0) + \psi(h_0) - \langle g_0, h_0 \rangle. \tag{50}$$

Also,

$$\Gamma_{\varphi,\psi}(g, h_0) = \varphi(g) + \psi(h_0) - \langle g, h_0 \rangle, \tag{51}$$
$$\Gamma_{\varphi,\psi}(g_0, h) = \varphi(g_0) + \psi(h) - \langle g_0, h \rangle. \tag{52}$$

Taking the difference between (51) and (50) gives (48). Taking the difference between (52) and (50) gives (49). □

Claim A.3. *Let $\mathcal{G} \subseteq \mathbb{R}^L$ be a bounded convex set. For $\lambda > 0$, if $\varphi : \mathcal{G} \to \mathbb{R}$ is a closed $\frac{1}{\lambda}$-strongly convex function w.r.t a norm $\|\cdot\|$ on $\mathbb{R}^L$, then its convex conjugate ψ is λ-smooth w.r.t. the dual norm $\|\cdot\|_*$.*

Proof. Consider an arbitrary $h_0 \in \mathbb{R}^L$ and let $g_0 \in \mathcal{G}$ be a subgradient of ψ at h_0. Since φ is closed, we know that (g_0, h_0) is a conjugate pair. By (48), the strong convexity of φ implies that

$$\Gamma(g, h_0) \geq \frac{1}{2\lambda}\|g - g_0\|^2 \qquad \text{for every } g \in \mathcal{G},$$
$$\text{or equivalently,} \quad \Gamma(g + g_0, h_0) \geq \frac{1}{2\lambda}\|g\|^2 \qquad \text{for every } g \text{ satisfying } g + g_0 \in \mathcal{G}.$$

It is easy to verify that the convex conjugate of $\Gamma(g + g_0, h_0)$ (as a function of g) is $\Gamma(g_0, h_0 + h)$ (as a function of h). It is also straightforward to see that the convex conjugate of $\frac{1}{2\lambda}\|g\|^2$ is $\frac{\lambda}{2}\|h\|_*^2$. Therefore, taking the convex conjugate of both sides of the inequality above, we have

$$\Gamma(g_0, h_0 + h) \leq \frac{\lambda}{2}\|h\|_*^2 \qquad \text{for every } h \in \mathbb{R}^L,$$
$$\text{or equivalently,} \quad \Gamma(g_0, h) \leq \frac{\lambda}{2}\|h - h_0\|_*^2 \qquad \text{for every } h \in \mathbb{R}^L.$$

By (49), this proves that ψ is λ-smooth w.r.t. $\|\cdot\|_*$. □

Proof of Lemma 2.3. Consider the negative Shannon entropy φ in (25). For $g, g_0 \in \Delta_L$, let $h_0 := \nabla\varphi(g_0)$. By (48) and Pinsker's inequality, we have

$$\varphi(g) - \varphi(g_0) - \langle g - g_0, h_0\rangle = \Gamma_{\varphi,\psi}(g, h_0) = \mathsf{D}_\varphi(g\|g_0) \geq \frac{1}{2}\|g - g_0\|_1^2.$$

This shows that φ is 1-strongly convex w.r.t. $\|\cdot\|_1$. By Claim A.3, we know that ψ is 1-smooth w.r.t. $\|\cdot\|_\infty$. Let $(g, h), (\hat{g}, \hat{h}) \in \mathcal{G} \times \mathbb{R}^L$ be two arbitrary conjugate pairs. By (48) and the 1-strong convexity of φ w.r.t. $\|\cdot\|_1$, we have

$$\Gamma(g, \hat{h}) = \varphi(g) - \varphi(\hat{g}) - \langle g - \hat{g}, \hat{h}\rangle \geq \frac{1}{2}\|g - \hat{g}\|_1^2.$$

Similarly, by (49) and the 1-smoothness of ψ w.r.t. $\|\cdot\|_\infty$, we have

$$\Gamma(g, \hat{h}) \leq \frac{1}{2}\|h - \hat{h}\|_\infty^2.$$

Combining the two inequalities above, we have

$$\|g - \hat{g}\|_1 \leq \|h - \hat{h}\|_\infty.$$

Since this holds for arbitrary $h, \hat{h} \in \mathbb{R}^L$ and the gradients $g := \nabla\psi(h) = \mathsf{softmax}(h), \hat{g} := \nabla\psi(\hat{h}) = \mathsf{softmax}(\hat{h})$, we have proved the 1-Lipschitzness of softmax. □