

# Polyadic encryption

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A novel original procedure of encryption/decryption based on the polyadic algebraic structures and on signal processing methods is proposed. First, we use signals with integer amplitudes to send information. Then we use polyadic techniques to transfer the plaintext into series of special integers. The receiver restores the plaintext using special rules and systems of equations.

*Keywords:* polyadic ring,  $n$ -ary group, signal processing, encryption, decryption, plaintext.

## I. INTRODUCTION

We propose a new approach to transfer hidden information in (continuous-time discrete-valued) signal processing (see, e.g. [1, 2]) by considering the parameters of signals not as the ordinary integers [3, 4], but as a special kind of integer numbers, polyadic integers, introduced in [5, 6]. The polyadic integers form polyadic  $(m, n)$ -rings having (or closed with respect to)  $m$  additions and  $n$  multiplications [7]. In this way, preservation of the property to be in the same polyadic ring after signal processing will give various restrictions on the signal parameters. The main our idea is to use these restrictions (as equations) to encrypt/decrypt series of ordinary numbers using sets of signals prepared in the special ways.

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## II. POLYADIC RINGS

We first remind that the polyadic ring or  $(m, n)$ -ring is a set with  $m$ -ary addition (being a  $m$ -ary group) and  $n$ -ary multiplication (being a  $n$ -ary semigroup), which are connected by the polyadic distributive law [7]. A polyadic ring is nonderived, if its full operations cannot be obtained as repetition of binary operations. As the simplest example of the binary ring or  $(2, 2)$ -ring is the set ordinary integers  $\mathbb{Z}$ , the example of  $(m, n)$ -ring is the set of polyadic integers  $\mathbb{Z}_{(m,n)}$  [5, 8]. The concrete realization of polyadic integers is the set of representative of the congruence class (residue class) of an integer  $a$  modulo  $b$  (with both  $a$  and  $b$  fixed)

$$\mathbb{Z}_{(m,n)}^{[a,b]} = [[a]]_b = \{\{a + b \cdot k\} \mid k \in \mathbb{Z}, a \in \mathbb{Z}_+, b \in \mathbb{N}, 0 \leq a \leq b - 1\}. \quad (2.1)$$

We denote a representative element by  $x_k = x_k^{[a,b]} = a + b \cdot k$ , which are polyadic integers  $\{x_k^{[a,b]}\} \in \mathbb{Z}_{(m,n)}^{[a,b]}$ , because only  $m$  additions and  $n$  multiplications are possible in the nonderived case

$$\overbrace{x_{k_1} + x_{k_2} + \dots + x_{k_m}}^m \in \mathbb{G}_{add(m)}^{[a,b]} \subset \mathbb{Z}_{(m,n)}^{[a,b]}, \quad (2.2)$$

$$\overbrace{x_{k_1} x_{k_2} \dots x_{k_n}}^n \in \mathbb{S}_{mult(n)}^{[a,b]} \subset \mathbb{Z}_{(m,n)}^{[a,b]}, \quad k_i \in \mathbb{Z}, \quad (2.3)$$

where  $\mathbb{G}_{add(m)}^{[a,b]}$  and  $\mathbb{S}_{mult(n)}^{[a,b]}$  are the  $m$ -ary additive group and  $n$ -ary multiplicative semigroup of the polyadic ring  $\mathbb{Z}_{(m,n)}^{[a,b]}$ .

It follows from (2.2)–(2.3), that more generally the  $m$ -admissible sum consists of  $\ell_m(m-1)+1$  summands and the  $n$ -admissible product contains  $\ell_n(n-1)+1$  elements, where  $\ell_m$  is a number of  $m$ -ary additions ( $m$ -polyadic power) and  $\ell_n$  is a number of  $n$ -ary multiplications ( $n$ -polyadic power). Therefore, in general

$$\overbrace{x_{k_1} + x_{k_2} + \dots + x_{k_m}}^{\ell_m(m-1)+1} \in \mathbb{G}_{add(m)}^{[a,b]} \subset \mathbb{Z}_{(m,n)}^{[a,b]}, \quad (2.4)$$

$$\overbrace{x_{k_1} x_{k_2} \dots x_{k_n}}^{\ell_n(n-1)+1} \in \mathbb{S}_{mult(n)}^{[a,b]} \subset \mathbb{Z}_{(m,n)}^{[a,b]}, \quad k_i \in \mathbb{Z}. \quad (2.5)$$

$\ell_m(m-1)+1$

For instance, in the residue (congruence) class

$$[[3]]_4 = \{\dots - 25, -21, -17, -13, -9, -5, -1, 3, 7, 11, 15, 19, 23, 27, 31, 35, 39 \dots\} \quad (2.6)$$

we can add  $4\ell_m + 1$  representatives and multiply  $2\ell_n + 1$  representatives ( $\ell_m, \ell_n$  are polyadic powers) to retain in the same class  $[[3]]_4$ . If, for example, we take  $\ell_m = 2$ ,  $\ell_n = 3$ , then we

get closeness of polyadic operations

$$(7 + 11 + 15 + 19 + 23) - 5 - 9 - 13 - 1 = 47 = 3 + 4 \cdot 11 \in [[3]]_4, \quad (2.7)$$

$$((7 \cdot 3 \cdot 11) \cdot 19 \cdot 15) \cdot 31 \cdot 27 = 55\,103\,895 = 3 + 4 \cdot 13\,775\,973 \in [[3]]_4. \quad (2.8)$$

Thus, we cannot add and multiply arbitrary quantities of representatives in  $[[3]]_4$ , only the admissible ones. This means that  $[[3]]_4$  is really the polyadic  $(5, 3)$ -ring  $\mathbb{Z}_{(5,3)}^{[3,4]}$ .

In general, a congruence class  $[[a]]_b$  is a polyadic ring  $\mathbb{Z}_{(m,n)}^{[a,b]}$ , if the following relations hold valid [5, 8]

$$(m-1) \frac{a}{b} = I^{(m)}(a, b) = I = \text{integer}, \quad (2.9)$$

$$\frac{a^n - a}{b} = J^{(n)}(a, b) = J = \text{integer}, \quad (2.10)$$

where  $I, J$  are called a (polyadic) shape invariants of the congruence class  $[[a]]_b$ . E.g., for the congruence class  $[[3]]_4$  the shape invariants (2.9)–(2.10) are  $I = 3$  and  $J = 6$ , correspondingly.

In the TABLE I the mapping

$$\Phi_{(m,n)}^{[a,b]} : (a, b) \longrightarrow (m, n) \quad (2.11)$$

of the congruence class parameters to the polyadic ring arities (we call it the arity shape) and shape invariants is presented for their lowest values. The arity shape mapping (2.11) is injective and non-surjective (empty cells), and it cannot be presented in the form of closed formula. Moreover, e.g., the congruence classes  $[[2]]_4$ ,  $[[2]]_8$ ,  $[[3]]_9$ ,  $[[4]]_8$ ,  $[[6]]_8$  and  $[[6]]_9$  do not correspond to any ring, while the same  $(6, 5)$ -ring can be described by different congruence classes  $[[2]]_5$ ,  $[[3]]_5$ ,  $[[2]]_{10}$ , and  $[[8]]_{10}$ .

The polyadic arity shape  $\Phi_{(m,n)}^{[a,b]}$  (2.11) is the main tool in the encryption/decryption procedure, described below.

### III. POLYADIC ENCRYPTION/DECRYPTION PROCEDURE

Let us consider the initial plaintext as a series of ordinary integer numbers (any plaintext can be transformed to that by the corresponding encoding procedure)

$$\mathbf{T} = y_1, y_2, \dots, y_r, \quad y_j \in \mathbb{Z}. \quad (3.1)$$

We propose a general encryption/decryption procedure, when each of  $y = y_j$  is connected with the various parameters of signal series, and the latter are transferred to the receiver, who then restores  $y$  using special rules and systems of equations known to him only.

TABLE I. The polyadic ring  $\mathbb{Z}_{(m,n)}^{[a,b]}$  of the fixed residue class  $[[a]]_b$ : the arity shape  $\Phi_{(m,n)}^{[a,b]}$ .

$a \setminus b$	2	3	4	5	6	7	8	9	10
1	$m = 3$ $n = 2$ $I = 1$ $J = 0$	$m = 4$ $n = 2$ $I = 1$ $J = 0$	$m = 5$ $n = 2$ $I = 1$ $J = 0$	$m = 6$ $n = 2$ $I = 1$ $J = 0$	$m = 7$ $n = 2$ $I = 1$ $J = 0$	$m = 8$ $n = 2$ $I = 1$ $J = 0$	$m = 9$ $n = 2$ $I = 1$ $J = 0$	$m = 10$ $n = 2$ $I = 1$ $J = 0$	$m = 11$ $n = 2$ $I = 1$ $J = 0$
2		$m = 4$ $n = 3$ $I = 2$ $J = 2$		$m = 6$ $n = 5$ $I = 2$ $J = 6$	$m = 4$ $n = 3$ $I = 1$ $J = 1$	$m = 8$ $n = 4$ $I = 2$ $J = 2$		$m = 10$ $n = 7$ $I = 2$ $J = 14$	$m = 6$ $n = 5$ $I = 1$ $J = 3$
3			$m = 5$ $n = 3$ $I = 3$ $J = 6$	$m = 6$ $n = 5$ $I = 3$ $J = 48$	$m = 3$ $n = 2$ $I = 1$ $J = 1$	$m = 8$ $n = 7$ $I = 3$ $J = 312$	$m = 9$ $n = 3$ $I = 3$ $J = 3$		$m = 11$ $n = 5$ $I = 3$ $J = 24$
4				$m = 6$ $n = 3$ $I = 4$ $J = 12$	$m = 4$ $n = 2$ $I = 2$ $J = 2$	$m = 8$ $n = 4$ $I = 4$ $J = 36$		$m = 10$ $n = 4$ $I = 4$ $J = 28$	$m = 6$ $n = 3$ $I = 2$ $J = 6$
5					$m = 7$ $n = 3$ $I = 5$ $J = 20$	$m = 8$ $n = 7$ $I = 5$ $J = 11160$	$m = 9$ $n = 3$ $I = 5$ $J = 15$	$m = 10$ $n = 7$ $I = 5$ $J = 8680$	$m = 3$ $n = 2$ $I = 1$ $J = 2$
6						$m = 8$ $n = 3$ $I = 6$ $J = 30$			$m = 6$ $n = 2$ $I = 3$ $J = 3$
7							$m = 9$ $n = 3$ $I = 7$ $J = 42$	$m = 10$ $n = 4$ $I = 7$ $J = 266$	$m = 11$ $n = 5$ $I = 7$ $J = 1680$
8								$m = 10$ $n = 3$ $I = 8$ $J = 56$	$m = 6$ $n = 5$ $I = 4$ $J = 3276$
9									$m = 11$ $n = 3$ $I = 9$ $J = 72$

The main idea is to examine such signals which have parameters as polyadics integers, that is they are in the polyadic ring  $\mathbb{Z}_{(m,n)}^{[a,b]}$  (2.1). This can be treated as a polyadic generalization of the (binary) discretization technique (in which the parameters are ordinary integers  $\mathbb{Z}$ ), and so we call it the polyadic discretization. Its crucial new feature is the possibility to transfer information (e.g. arities, minimal allowed number of additions and multiplications) using signal parameters, as it will be shown below.

Here we apply this idea to signal amplitudes (such signal is called the continuous-time discrete-valued or quantized analog signal [9, 10]) and their addition only. This means that

we look on the additive part of the polyadic ring  $\mathbb{Z}_{(m,n)}^{[a,b]}$  which is a nonderived (minimum only  $m$  terms can be added)  $m$ -ary group  $\mathbb{G}_{add(m)}^{[a,b]}$  (2.2). In this way we denote the single  $i$ th signal shape as

$$\Psi_i^{(\ell_f)} = A_i^{(\ell_f)} \cdot f^{(\ell_f)}(t), \quad (3.2)$$

where  $A_i^{(\ell_f)}$  is the amplitude of the normalized (in some manner)  $i$ th signal  $f^{(\ell_f)}(t)$ ,  $t$  is time, and the natural  $\ell_f \in \mathbb{N}$  corresponds to the special kind of signal (by consequent numerating sine/cosine, triangular, rectangular, etc.).

First, we assume that the amplitude  $A_i^{(\ell_f)}$  is in polyadic ring, i.e. it is a representative of the congruence class  $[[a]]_b$ ,  $b \in \mathbb{N}$ ,  $0 \leq a \leq b - 1$ , and therefore it has the form (2.1)

$$A_i^{(\ell_f)} = a + b \cdot k_i^{(\ell_f)} \in \mathbb{G}_{add(m)}^{[a,b]} \subset \mathbb{Z}_{(m,n)}^{[a,b]}, \quad k_i^{(\ell_f)} \in \mathbb{Z}. \quad (3.3)$$

Second, we identify the number of signal species  $\ell_f$  with the  $m$ -polyadic power  $\ell_m$  from (2.4)

$$\ell_f = \ell_m. \quad (3.4)$$

In this picture, for the signal species  $\ell_f$  we prepare the sum of  $\ell_f(m - 1) + 1$  signals as

$$\Psi_{tot}^{(\ell_f)} = \sum_{i=1}^{\ell_f(m-1)+1} \Psi_i^{(\ell_f)} = A_{tot}^{(\ell_f)} \cdot f^{(\ell_f)}(t) \quad (3.5)$$

where the total amplitude  $A_{tot}^{(\ell_f)}$  becomes different for distinct species  $\ell_f$  and after usage of (3.3) has the general form

$$A_{tot}^{(\ell_f)} = \sum_{i=1}^{\ell_f(m-1)+1} A_i^{(\ell_f)} = a \cdot (\ell_f(m - 1) + 1) + b \cdot K(m, \ell_f) \in \mathbb{G}_{add(m)}^{[a,b]} \subset \mathbb{Z}_{(m,n)}^{[a,b]}, \quad (3.6)$$

$$K(m, \ell_f) = \sum_{i=1}^{\ell_f(m-1)+1} k_i^{(\ell_f)}. \quad (3.7)$$

Thus, we observe that the total amplitude (3.6) of the signal (3.2) contains all parameters of the  $m$ -ary group  $\mathbb{G}_{add(m)}^{[a,b]}$  (the additive part of the polyadic ring  $\mathbb{Z}_{(m,n)}^{[a,b]}$  (2.1)). This allows us to use the combination of signals (in the above particular case sums) to transfer securely the plaintext variables  $y_j$  (3.1) from sender to receipient, if we encode each of them  $y = y_j$  by the polyadic ring parameters  $y \rightarrow (a, b, m)$ . The receipient obtains the set of the total amplitudes  $A_{tot}^{(\ell_f)}$  and treats (3.6) as the system of equations for parameters  $(a, b, m)$ , and

then after the decoding  $(a, b, m) \longrightarrow y \longrightarrow y = y_j$  for each  $j$  obtains the initial plaintext  $\mathbf{T}$  (3.1). Schematically, we can present the proposed encryption/decryption procedure as

$$\text{snd: } y \xrightarrow{\text{encoding by snd}} (a, b, m) \xrightarrow{\text{summing}} A_{tot}^{(\ell_f)} \xrightarrow{\text{preparing signals}} \Psi_{tot}^{(\ell_f)} \xrightarrow{\text{transferring to rcp}} \quad (3.8)$$

$$\text{rcp: system of equations } A_{tot}^{(\ell_f)} \text{ (3.6)} \xrightarrow{\text{solving by rcp}} (a, b, m) \xrightarrow{\text{decoding by rcp}} y \quad (3.9)$$

The security of this procedure is goverened not by one key, as in the standard cases, but by the system (3.6) and the TABLE I, as well as by the correspondence (numbering) between kind of signal  $\ell_f$  and  $m$ -polyadic power (3.4), which are all unknown to the third party.

#### IV. EXAMPLE

Let us consider a concrete example of the proposed encryption/decryption procedure for the congruence class  $[[a]]_b$  and one kind of signal  $\ell_f = \ell_m \equiv \ell$ , where  $\ell$  is  $m$ -polyadic power. Each such class gives (by TABLE I) the arity  $m$  as the plaintext entry to transfer  $y = m$ . Next we should choice the shape of the function  $K(m, \ell)$  (3.7), which is, in general, arbitrary. In the simplest case, we take the same linear function

$$k_i^{(\ell)} = i - 1, \quad (4.1)$$

for all  $m$ -polyadic powers  $\ell$ , but any any functional dependence in (4.1) can be chosen and different for different  $\ell$ , which increases security of the procedure considerably. The choice (4.1) gives

$$K(m, \ell) = \frac{1}{2} \ell (m - 1) (\ell (m - 1) + 1). \quad (4.2)$$

So the total amplitudes for different polyadic powers become

$$A_{tot}^{(\ell)} \equiv B_\ell = a \cdot (\ell (m - 1) + 1) + \frac{b}{2} \ell (m - 1) (\ell (m - 1) + 1). \quad (4.3)$$

The receipient obtains the set of signals with amplitudes (4.3) as polyadic integers

$$\Psi_{tot}^{(\ell)} = B_\ell \cdot f^{(\ell)}(t). \quad (4.4)$$

To obtain the values of three variables  $(a, b, m)$ , one needs 3 equations, i.e. 3 total amplitues with different arbitrary polyadic powers  $\ell = \ell_1, \ell_2, \ell_3 \in \mathbb{N}$ . Because the general solution is too cumbersome, we choose the first 3 consequent polyadic powers  $\ell = 1, 2, 3$ ,

while any three natural numbers are possible to increase the security. This gives the following system of quadratic equations

$$a \cdot m + b \cdot \frac{(m-1)m}{2} = B_1, \quad (4.5)$$

$$a \cdot (2m-1) + b \cdot (m-1)(2m-1) = B_2, \quad (4.6)$$

$$a \cdot (3m-2) + b \cdot \frac{3(m-1)(3m-2)}{2} = B_3. \quad (4.7)$$

The general solution of the system is

$$m = \frac{7B_1 - 4B_2 + B_3 \mp \sqrt{B_1^2 - 8B_2B_1 - 2B_3B_1 + 16B_2^2 + B_3^2 - 8B_2B_3}}{4(3B_1 - 3B_2 + B_3)}, \quad (4.8)$$

$$a = 3B_1 - 3B_2 + B_3, \quad (4.9)$$

$$\begin{aligned} b = & \frac{1}{(3B_1 - 3B_2 + B_3)(2B_2 - B_1 - B_3)} \times \\ & \left[ (11B_1^2 - 16B_2B_1 + 6B_3B_1 - 4B_2^2 + 4B_2B_3 - B_3^2)(3B_1 - 3B_2 + B_3) \right. \\ & - \frac{105}{2}B_1^3 + \frac{333}{2}B_1^2B_2 - 162B_1B_2^2 + 48B_2^3 - \frac{113}{2}B_1^2B_3 \\ & + 107B_1B_2B_3 - 46B_2^2B_3 - \frac{35}{2}B_1B_3^2 + \frac{29}{2}B_2B_3^2 - \frac{3}{2}B_3^3 \\ & \left. \pm \left( \frac{15}{2}B_1^2 - \frac{39}{2}B_1B_2 + 12B_2^2 + 7B_1B_3 - \frac{17}{2}B_2B_3 + \frac{3}{2}B_3^2 \right) \times \right. \\ & \left. \sqrt{B_1^2 - 8B_2B_1 - 2B_3B_1 + 16B_2^2 + B_3^2 - 8B_2B_3} \right]. \quad (4.10) \end{aligned}$$

The sign in (4.8) and (4.10) should be chosen so that the solutions are ordinary integers.

In particular case, for instance, the congruence class  $[[3]]_4$  (2.6), has 5-ary addition, as follows from TABLE I. So if the sender wants to submit securely one element  $y = m = 5$  from his plaintext (3.1), he applies the proposed encryption procedure and prepares three sums of (quantized analog) signals (having integer amplitudes) corresponding to the polyadic powers  $\ell = 1, 2, 3$ , as follows (using (3.5) and (4.3))

$$\Psi_{tot}^{(1)} = 55 \cdot f^{(1)}(t), \quad (4.11)$$

$$\Psi_{tot}^{(2)} = 171 \cdot f^{(2)}(t), \quad (4.12)$$

$$\Psi_{tot}^{(3)} = 351 \cdot f^{(3)}(t), \quad (4.13)$$

where  $f^{(1,2,3)}(t)$  are different (or the same) normalized signals. The recepient receives 3 (quantized analog) signals (4.11)–(4.13), and because he knows the normalized signals, he

immediately obtains the integer values  $B_1 = 55$ ,  $B_2 = 171$ ,  $B_3 = 351$ . Inserting them into the system of quadratic equations (known to him ahead) (4.5)–(4.7), he (directly or using (4.8)–(4.10)) derives the values  $m = 5$ ,  $a = 3$ ,  $b = 4$  and the desired element  $y = m = 5$  from the initial plaintext (3.1). The same procedure should be provided for each element  $y_j = y$  of the plaintext (3.1), which completes its decryption.

## V. CONCLUSIONS

Thus, we proposed a principally new encryption/decryption procedure based on exploiting the signal processing. The main idea is to consider the signal parameters as polyadic integers being representative of the fixed polyadic ring, which is treated as a some kind of polyadic discretization depending of the ring integer characteristics. They allow us to transfer numerical information from sender to receipient by submitting special sets of signals with genuine properties agreed before. The receipient knows the rules and equations to solve, to decrypt the initial plaintext. The security is achieved by seemed arbitrariness of the openly transfered signals and additional information which is impossible to identify by third party.



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- [1] A. V. Oppenheim, *Applications of Digital Signal Processing* (Prentice-Hall, Englewood Cliffs, N.J., 1978).
  - [2] B. Wang, *Digital Signal Processing Techniques and Applications in Radar Image Processing* (Wiley, Hoboken, NJ, 2008).
  - [3] S. Berber, *Discrete Communication Systems* (Oxford University Press, Oxford, 2021).
  - [4] A. Pedrouzo-Ulloa, J. R. Troncoso-Pastoriza, and F. Pérez-González, Number theoretic transforms for secure signal processing, *IEEE Trans. Inf. Forensics Secur.* **12**, 1125 (2017).
  - [5] S. Duplij, Polyadic integer numbers and finite  $(m, n)$ -fields, *p-Adic Numbers, Ultrametric Analysis and Appl.* **9**, 257 (2017).
  - [6] S. Duplij, *Polyadic Algebraic Structures* (IOP Publishing, Bristol, 2022) p. 461.
  - [7] J. J. Leeson and A. T. Butson, On the general theory of  $(m, n)$  rings., *Algebra Univers.* **11**, 42 (1980).
  - [8] S. Duplij, Arity shape of polyadic algebraic structures, *J. Math. Physics, Analysis, Geometry* **15**, 3 (2019).
  - [9] J. Musayev and A. Liscidini, Quantised inverter amplifier, *Electron. Lett* **54**, 416 (2018).
  - [10] A. Karthik, A. V, G. Nijhawan, A. Rana, I. Khan, and Z. L. Naser, Quantum signal processing: a new frontier in information processing, in *10th IEEE Uttar Pradesh Section International Conference on Electrical, Electronics and Computer Engineering (UPCON)*, Vol. 10 (IEEE, 2023) pp. 1527–1532.