Pseudo-Equilibria, or: How to Stop Worrying About Crypto and Just Analyze the Game

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Abstract

We consider the problem of a game theorist analyzing a game that uses cryptographic protocols. Ideally, the game theorist should be able to ignore all implementation details of the cryptographic protocols, and abstract them as ideal, implementation-independent primitives, in a way that conclusions in the "ideal world" can be faithfully transferred to the "real world," where real protocols are implemented by cryptography. Achieving this goal is crucial, as the game theorist cannot — and should not be expected to — grapple with the full complexity of cryptographic implementations. This is particularly relevant in the era of Web3, where the widespread adoption of distributed ledgers has created a pressing need for a common language that bridges cryptography and game theory.

The security of cryptographic protocols hinges on two types of assumptions: state-of-the-world assumptions (e.g., "one-way functions exist", "factoring is hard", etc.), and behavioral assumptions (e.g., "honest majority"). We observe that for cryptographic protocols that rely on behavioral assumptions (e.g., ledgers), our goal is unattainable in its full generality: the game theorist cannot always abstract away all implementation details. For state-of-the-world assumptions, we show that standard solution concepts, e.g., Nash equilibria and ϵ -Nash equilibria, are not robust to the transfer from the ideal setting to the real world.

In this paper, we propose a new solution concept: the *pseudo-Nash equilibrium*. Informally, a strategy profile $s = (s_1, \ldots, s_n)$ is a pseudo-Nash equilibrium if, for every player *i* and every unilateral deviation s'_i with a larger expected utility, the utility of *i* when playing s_i is (computationally) indistinguishable from her utility when playing s'_i . Pseudo-Nash is substantially simpler and more accessible to game theorists than any existing notion that attempted to address the mismatch in the (asymptotic) cryptographic method and game theory. We prove, in a very general sense, that Nash equilibria in games that use idealized, unbreakable cryptography correspond naturally to pseudo-Nash equilibria when idealized cryptography is instantiated with actual protocols (under state-of-the-world assumptions). Our translation is not only conceptually simpler than existing approaches, but also more general: it does not require tuning or restricting utility functions in the game with idealized cryptography to accommodate idiosyncrasies of cryptographic implementations. In other words, pseudo-Nash equilibria allow us to separately and seamlessly study game-theoretic and cryptographic aspects.

1 Introduction

Cryptography and game theory both study interactions among agents. Cryptography traditionally takes an adversarial approach; protection is demanded against the worst-case (mis)behavior of the agents. Game theory takes the view that agents are selfish and aim to optimize their own utility. The widespread adoption of blockchain protocols has set a stage where both viewpoints are invaluable, as decentralized systems must be both strategically robust and cryptographically secure.

However, while the interaction of the two fields has been fruitful, we believe it is fair to say that a universally adopted model that allows for the two fields to interact seamlessly has yet to emerge. Arguably, the most ambitious goal would be a model such that game theorists can work in an "ideal world" that completely ignores all implementation details of cryptographic protocols (i.e., treats cryptographic primitives as black boxes that provide a certain functionality), but such that game-theoretic solution concepts, e.g., equilibria, naturally carry over to the "real world," where cryptographic primitives are replaced by their (imperfect) implementations. Such a model, by design, would also allow cryptographers to implement cryptographic primitives without worrying about downstream effects on game theoretic predictions. Simply put, our work attempts to answer the following question: *Is this ambitious goal attainable*?

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1.1 Our contributions

Unfortunately, the answer to the aforementioned question is, in general, "no." This is because cryptographic protocols provide guarantees only under assumptions. There are, broadly speaking, two types of assumptions cryptographic protocols require. The first type is *state-of-world* assumptions; examples of this type are computational hardness, ideal obfuscation, or secure hardware, which are common assumptions in the design of cryptographic protocols like encryption, digital signatures, commitments, etc. The second type is *behavioral* assumptions (e.g., "honest majority"), which are assumptions about the parties that implement the protocol at hand. The reason we cannot have a model that seamlessly abstracts away all cryptographic details, but is faithful to game-theoretic solution concepts, is because of this second type. Many central solution concepts in game theory, e.g., Nash equilibria, are predictions about players' behavior. Therefore, intuitively, it is not possible to ignore elements of a game that only work under specific player behavior. For a concrete example, consider the following.

Example 1.1 (Games on a Ledger). Consider the problem of a miner on a blockchain. The blockchain is an implementation of cryptographic primitives known as distributed ledgers, which, loosely speaking, provide the following functionality: every party can provide inputs to the ledger which is recorded on a future block; every party can receive the ledger's current state.¹ The blockchain implements the ledger functionality assuming an honest majority of the participants, or of a resource held by the participants like hashing power (in proof of work) or stake (in proof of stake); the miners. Given an ideal ledger, consider a simple game between miners, where a miner gains a large reward R when doing a task quickly, and a small reward r when doing the task slowly, where "time" is measured by the timestamps on blocks. And, for the sake of exposition, assume that the miner pays a large cost when doing the task slowly, but no cost when doing the task slowly, and thus the miner's best response under these rules is to do the task slowly. In the context of DeFi (Decentralized finance), [YTZ22] prove that, in certain blockchain protocols, it is possible for miners to deviate in a way that they can manipulate the timestamps of blocks. For our simple game, this would mean that the miner could claim the large reward R, contradicting the prediction in the ideal game.

We note that, starting from [ES18], numerous works [BW24, BCNPW19, CKWN16, FHWY22, FGH⁺24, FW21, KKKT16, FKKP19, GS19, SSZ17, TE18] show how, under several different classes of blockchain protocols, there are several ways that miners can manipulate their behavior in the protocol to manipulate timestamps, their rewards in the underlying cryptocurrency, or even the value of the cryptocurrency itself. Thus, the existence of these counterexamples is well-known (but perhaps not articulated using the ideal vs real-world perspective we take in this work).

Since our goal is unattainable in general, the next natural question is whether it is attainable under conditions. And, given our discussion so far, a natural condition would be the usage of cryptographic protocols that make state-of-the-world assumptions. Our next observation is that standard game-theoretic solution concepts do not quite work.

Example 1.2 (The Guessing Game). Consider a simple, ideal two-player game, where player 1 commits to κ bits, and player 2 needs to guess these κ bits. If player 2 guesses all κ bits, then she wins and gets a utility of 1; otherwise, player 1 wins and player 2 gets a utility of -1. An ideal commitment scheme C, for the case of two players, works as follows. Player 1 sends a value x to C, and C sends a "receipt" c to player 2. Later, player 1 sends "open" to C, and C sends x to player 2. In an idealized world, C is perfectly binding (there is nothing player 1 can do to open the commitment C to any $x' \neq x$) and perfectly hiding (there is nothing player 2 can learn about x from c). It is easy to see that, when one uses an ideal commitment scheme in the game described, in the unique Nash equilibrium of the game, both players pick an κ -bit string uniformly at random (and player 2 wins with probability $1/2^{\kappa}$).

In the "real game," the commitment scheme can be implemented in various ways. Perfectly hiding and perfectly binding schemes are known to not exist, but, for example, the ElGamal-based commitment scheme is computationally hiding and perfectly binding. Concretely, a "real world" cryptographic commitment scheme consists of a function Commit(x,r) that takes as input a message x and a random value r, and outputs a commitment c; then "open" can be implemented by simply revealing (x,r). Computationally hiding then means that, for r and r' chosen uniformly at random, the distributions of Commit(x,r) and Commit(x',r') are computationally indistinguishable, i.e., no efficient algorithm can distinguish which one is which with probability negligibly better than a random cointoss. Importantly, the above indistinguishability statement does not exclude that an inefficient adversary, who, e.g.,

¹We note that we oversimplify the functionality, in order to make our argument. Examples of ledger functionalities implemented by mainstream blockchain protocols like Bitcoin and Ouroboros/Cardano—and corresponding proofs of security—can be found in [BMTZ17, BGK⁺18].

performs a brute-force attack by checking all values of x and r against c, can guess x with good probability. In fact, when the commitment scheme is perfectly binding, such a bruteforcing is always possible. Thus, player 2's best response to every strategy of player 1 is to attempt to guess (x, r) by brute-forcing. And, even imposing computational constraints on player 2, brute forcing will still be a strategy that succeeds with a probability δ better than random guessing. Therefore, picking a uniformly random κ -bit string is not a best-response for player 2. And, by making the rewards sufficiently large (e.g., infinite), it is not even an ϵ best-response, for any $\epsilon > 0$.

Even though Example 1.2 rules out the Nash equilibrium (and ϵ -Nash) as a general "plug-and-play" solution concept, it seems intuitively possible to fix the issue by slightly changing the rules. So, for example, we could "assume-away" the undesirable actions (players' attempts to break cryptography) in the real game of Example 1.2; then, we should be able to conclude that picking an κ -bit string uniformly at random (the Nash equilibrium of the ideal game) is a Nash equilibrium of the real game. However, assuming away cryptographic attacks is problematic—what does it formally mean to assume away in Example 1.2 that player 2 "attempts to guess"? Perhaps more troubling, as our next example shows, the transition from the ideal world to the real world can result in two games with completely different equilibria and approximate equilibria.

Example 1.3 (Auctions with Information Advantage). Consider running a sealed-bid second-price auction for selling a single item. Sealed bids are implemented by commitment schemes, similarly to Example 1.2. Bidder i's utility is their private value v_i minus the price they are asked to pay p_i . Additionally, bidder i gets utility by knowing the identity of the agent with the highest value; this scenario can occur, for example, if having information about who won the auction offers agents a competitive advantage in future auctions. Concretely, after the auction finishes, bidder i gets a large reward R with probability $q_i = \max\{x_{i,i^*} - \frac{1}{n-1}, 0\}$, where $x_{i,j}$ is the probability that i assigns to the event that agent j has the highest value (and therefore, q_i can be interpreted as how much better i can do, compared to a random guess).

Under ideal commitment schemes, it is easy to see that bidders will simply bid their true value in the auction, and then assign probabilities $x_{i,j} = \frac{1}{n-1}$ to all $j \neq i$ (except, of course, the winner of the auction i^* who has $x_{i^*,i^*} = 1$). In the real world, under a cryptographic implementation, a first observation is that every bidder is better off attempting to break cryptography after the auction ends, i.e., as in Example 1.2, i should attempt to break the commitment scheme and get a strict improvement in her expected utility. However, for a high enough reward R, an even more profitable strategy exists: submit an extremely high bid to win the auction, and therefore learn the true highest value v_{i^*} (since this is now the payment). Breaking the cryptographic commitment is now a lot easier: i only needs to learn which bidder committed to the value v_{i^*} (while earlier her task was to guess the value as well). Therefore, for large enough R, the Nash equilibrium of the real game has players bidding non-truthfully!

Examples 1.2 and 1.3 paint a bleak picture, at least for Nash and ϵ -Nash equilibria. Yet, in both cases, the Nash equilibria in the ideal game seem like the "correct" predictions of how agents will behave in the corresponding real games. But, if these strategies are not Nash or ϵ -Nash equilibria in the real game, then what are they? The main contribution of our paper is to propose a new solution concept — the *pseudo-Nash equilibrium* — that allows us to reconcile this gap.

Pseudo-equilibria. A strategy profile $s = (s_1, s_2, \ldots, s_n)$ is a Nash equilibrium if, for every agent *i*, the expected utility of *i* when playing strategy s_i is at least her utility when deviating to strategy s'_i (when all other agents' strategies remain fixed). The above definition makes no assumption about how efficient the strategies in *s* are. However, in games that use cryptography, it is important to exclude strategies that require inefficient (e.g., super-polynomial in the cryptographic security parameter) computation. Indeed, allowing players this amount of computation makes breaking cryptography a feasible strategy for them. A strawman approach to solve the above conundrum is to enforce that the strategies are polynomial-time computable. It is, however, not hard to verify that such a simple patch cannot be the solution: First, by doing so one necessarily changes the description of the game, resulting in restricted and often complicated stability definitions that require hard-wiring cryptographic security parameter. Such a choice might seem natural to a cryptographer but substantially dilutes our goal of delivering a stability notion, which is friendly to economists and game-theorists with little exposure to cryptographic definitions.

Second, and more importantly, the above modification does not even solve the problem. To see why, one can look at the Guessing Game (Example 1.2): Even for a polynomial bounded player, brute-forcing the cryptographic primitive is still best response. In fact, on a more technical note, one would need to actually bound the player's reasoning (or rationality) to be polynomial time. In order to do so, one would need to consider such reasoning as part of the player's strategy which is known to lead to cumbersome definitions. Another strawman approach, which takes care of the above issues, would be to define pseudo-equilibrium as a strategy whose utility profile is (computationally) indistinguishable from a real equilibrium. This, however, is also quite problematic: first, in classical equilibrium definitions, one is interested in expected utilities. However, two random variables that are computationally indistinguishable do not necessarily have the same expectation. For example, consider a game such that the utility random variables induced by two strategies are $U_1 = \{0 \text{ w.p. } 1 - \epsilon \& 1 \text{ w.p. } \epsilon\}$ and $U_2 = \{0 \text{ w.p. } 1 - \epsilon \& -1 \text{ w.p. } \epsilon\}$. These two random variables are computationally indistinguishable for negligible ϵ , but the expected value of U_1 is positive and the expected value of U_2 is negative. Furthermore, two random variables can have the same expected value but completely different support (e.g., $U_1 = \{1 \text{ w.p. } 1/2 \& -1 \text{ w.p. } 1/2\}$ and $U_2 = \{10 \text{ w.p. } 1/2 \& -10 \text{ w.p. } 1/2\}$). These would be equivalent in the eyes of a game theorist, but are clearly distinguishable.

Second, such a notion would only be defined in games where their idealized-crypto variant can be shown to have a Nash equilibrium. This approach, which is implicit in [HP19], substantially reduces the applicability of pseudo-equilibrium and makes the notion applicable only with respect to such games, which again makes it less accessible to non crypto-savvy domain experts.

In this work, we take a different approach. First, we observe that, intuitively, a necessary condition for strange strategies not to emerge as profitable deviations in any game (e.g., the "real game" discussed above) is that events that occur with negligible probability (e.g., "break crypto") should be ignored. In other words, the condition "the expected utility of the deviation should not be larger" in the definition of Nash should be replaced; we should instead focus on different (statistical) properties of the utilities of strategies. Our definition of a pseudo-equilibrium formalizes this intuition.

We begin by attempting to replace "larger expectation" in the definition of Nash, with the concept of indistinguishability from cryptography. We have that two random variables X and Y are ϵ -indistinguishable, if, for any distinguisher \mathcal{D} , $|\Pr[\mathcal{D}(X) = 1] - \Pr[\mathcal{D}(Y) = 1]| \leq \epsilon$. A first observation is that in the context of a game, where the random variables X and Y naturally correspond to utilities of strategies (e.g. a strategy and a deviation), this definition overlooks a crucial point: if a strategy s is a lot better than a strategy s', we should be able to distinguish them!

A straightforward remedy for this issue is to drop the absolute value from the previous definition. Adjusting to our game-theoretic goals, perhaps we should define a strategy s, whose utility is distributed according to a random variable X, to be " ϵ -preferred" to a strategy s', whose utility is distributed according to a random variable Y, if, for any distinguisher \mathcal{D} , $\Pr[\mathcal{D}(Y) = 1] - \Pr[\mathcal{D}(X) = 1] \leq \epsilon$. Under this definition, when X is a lot "better" than Y, we allow for distinguishers that pick X a lot more often. However, this definition also has an issue, albeit a more nuanced one.

Concretely requiring the above for all distinguishers, implies requiring it for the best one, i.e., the one that maximizes $\Pr[\mathcal{D}(Y) = 1] - \Pr[\mathcal{D}(X) = 1]$. Intuitively, \mathcal{D} can be seen as implementing the best statistical test. However, the outcome of this best distinguisher might have a weak (or no) dependence on the expectations of X and Y. This makes it challenging (and in some cases, impossible) to connect a notion based on this distinguisher to standard game theory concepts that only look at the expectation of (the outcome) of strategies. As a result, the corresponding equilibrium notion would not necessarily be implied by Nash equilibrium in standard games, which takes us away from our goal to find a version of Nash (satisfying the same intuitive stability properties) that is cryptography-friendly.

Nonetheless, the above intuition is on the right track. The key is to identify a statistical test (i.e., distinguisher) whose decision approximately follows the relation of the expectations, i.e., the test should favor strategy X when the expectation of X is greater than that of Y. This brings us to our (informal) definition of a pseudo-equilibrium, where instead of considering arbitrary distinguishers, we focus on the simple distinguisher that compares the sample means of the two random variables:

Definition 1.1 (Pseudo-Nash (informal)). Let $s = (s_1, s_2, ..., s_n)$ be a strategy profile, and let $U_i(s; s_{-i})$ be the random variable that indicates the utility of player *i* under strategy *s*, when all other players are playing according to strategies in s_{-i} . Then, *s* is an (m, ϵ) -pseudo Nash equilibrium, if for all players *i*, and strategies *s'*

$$\Pr\left[\frac{1}{m}\sum_{j=1}^{m}U_{i}^{(j)}(s';s_{-i}) \ge \frac{1}{m}\sum_{j=1}^{m}U_{i}^{(j)}(s_{i};s_{-i})\right] - \Pr\left[\frac{1}{m}\sum_{j=1}^{m}U_{i}^{(j)}(s_{i};s_{-i}) \ge \frac{1}{m}\sum_{j=1}^{m}U_{i}^{(j)}(s';s_{-i})\right] \le \epsilon,$$

where $U_i^{(j)}(s; s_{-i})s$ are *i.i.d.* samples from $U_i(s; s_{-i})$.

The above definition is still informal, and several refinements are needed to make it compatible with cryptography. First, in order to use cryptographic reasoning we need the distinguishing bias ϵ to be *negligible*—in

cryptography, a function of n is negligible if diminishing faster than the inverse of any polynomial in n (see Definition 2.1). Second, in order to allow the above to hold for cryptographic distinguishers, we need to allow the number of samples to be polynomial (in the size of the game and or a security parameter). The formal definition (Definition 3.2) takes all the above into consideration and achieves all the desired properties:

- 1. Simplicity: Our notion does not require any knowledge from cryptography, and can be defined for any game. This is in contrast to state-of-the-art proposals for crypto-friendly equilibria, e.g., [HPS16] which require special, complicated, classes of games. See Section 1.2 and Appendix B for more details.
- 2. *Compatibility with Game Theory*: In standard non-parameterized games, the notion of a pseudo-Nash equilibrium is equivalent to the notion of a Nash equilibrium.
- 3. *Indistinguishability*: The replacement of random variables with computationally indistinguishable ones does not affect our notion of dominance.
- 4. Unrestricted utilities: Unlike many existing attempts to devise notion of equilibrium for computational games which require restrictions on the utility, (e.g., [HPS16] assumes utilities of low probability events which are polynomial in a parameter of the game) a pseudo-equilibrium is not sensitive to the magnitude of the utility of low probability events.

In the technical part of this paper, we prove the above properties, and present examples from the literature along with simple games, which showcase our notion and highlight its benefits over prior proposals. Most importantly, for the natural class of normal-form games that corresponds to games using cryptography—we call these *computational games*—we can prove a general ideal-crypto-replacement theorem:

Theorem 1.1 (Informal). Consider any (ideal) computation game $\mathcal{G}^{\mathcal{I}}$ using ideal cryptographic primitives. Consider the (real) computation game $\mathcal{G}^{\mathcal{R}}$ where the ideal primitive is replaced by its cryptographic implementation—secure under a state-of-the-world assumption. If a strategy $s^{\mathcal{I}}$ is a Nash or pseudo-Nash equilibrium in $\mathcal{G}^{\mathcal{I}}$, then the corresponding strategy $s^{\mathcal{R}}$ —where ideal cryptography is replaced with ideal cryptography—is a pseudo-Nash equilibrium in $\mathcal{G}^{\mathcal{R}}$.

We note in passing that in addition to having all the above desirable properties, pseudo-Nash has also a rather intuitive interpretation, which reinforces our claim that it is the natural notion for the games we aim to capture, and beyond: Recall that we are interested in equilibrium for players whose reasoning is computationally limited. The definition of pseudo-Nash captures the perception—from the point of view of a player who can only think of up to *m* iterations of the game—of whether or not deviating from s_i dominates s_i (within some error ϵ .) We thus conjecture that our thinking might have connections to prospect theory [KT79, TK92]; investigating such connections is, in our opinion, an outstanding research direction.

1.2 Related Work

A number of works have attempted to use game theory within cryptographic protocols (typically MPC or its special cases, e.g., [DHR00, HT04, GK06, FKN10, LMS05, IML05, KN08]). The idea of such *game-theoretic cryptography* models is to incorporate incentives into the parties' misbehavior by treating them as rational agents who act according to some (partially) known preferences. A cryptographic protocol execution is thus viewed as a sequential game among these agents/parties and is considered "secure" if they have no incentive to deviate, i.e., if it induces a game-theoretic equilibrium. Notably, several of these works have highlighted mismatches between the tools used to reason in the two disciplines. For example, in [KN08] it was shown that under standard equilibrium definitions cryptography cannot be used in certain games in which privacy is important for the utility, e.g., in games where selfish parties want to learn a common function on their input but everyone prefers to be the only one who learns it. Most interesting, a frequent reason for impossibilities has been rooted to the idea that it might be best response for the adversary to try to break cryptography.

Similarly, several works have used cryptography primitives to improve game theory and mechanism design solutions (e.g., [NS93, BPRP08, FW20, EFW22, TAF⁺23]), e.g., by relying on cryptographic commitments to force consistency between different stages of the game.

Most relevant to our goals, several works have attempted to resolve mismatches between cryptography and game theory [GLR13, HPS16, HP16, HPS14, GKM⁺13, HP15]. The standard approach in this area has been to either adapt the utility to account for computational complexity and asymptotics, e.g., by allocating a cost on computational steps [HP15], or develop new definitions of stability grounded in cryptography. Such definitions

typically change the nature of the game, e.g., by transitioning from a simple, single game to a parameterized sequence of (usually more complex) games, in order to accommodate asymptotic reasoning, and define an equilibrium notion on top of this sequence. As such, they have typically been much more cumbersome compared to their (arguably simple and intuitive) game theoretic analogues, such as Nash or subgame-perfect equilibrium. We conjecture that this might be a reason why these new definitions, despite their evident novelty—and even support for replacing ideal cryptography with its cryptographic implementation [HP15, HPS14, GKM⁺13]—have not been widely adopted by economists or game theorists, who are not necessarily familiar with theoretical computer science and cryptography idiosyncrasies. In contrast, our goal is to give such researchers a convenient tool to incorporate cryptography in their game analysis, without worrying about the aforementioned artifacts.

In order to understand the novelty and potential of the notion of pseudo-equilibrium it is worth contrasting it with the notion of *computational Nash* in extensive form games [HPS16], which, to the best of our knowledge, is the state-of-the-art among equilibrium definitions that allow using cryptography within game-theory arguments.² For readers unfamiliar with this notion we have included its definition in Appendix B. In a nutshell, the notion of computational Nash [HPS16] defines a new class of (sequences of) games, termed Computable Uniform Sequence of Games, which embodies the idea that cryptographic protocols can be viewed as extensive-form games, where adversarial faults correspond to deviations. In such a game definition, one needs to carefully define properties of the history, utility, and players' moves to ensure the are polynomially computable—in fact, probabilistic polynomialtime, in short PPT (see Definition B.1). As discussed below, this makes the notion of computational Nash applicable only to games that fit such a description. The above complexity makes analyzing computational Nash (see Definition B.2) directly a particularly challenging task. In fact, this is acknowledged in [HPS16], which offers a blueprint for finding computational Nash, which as will become apparent below, can only be used when we are analyzing a game that is derived by replacing idealized cryptography in a sequential game with real cryptography. The core idea of this methodology is to "reverse-engineer" the cryptographic simulation paradigm to establish a mapping of the game using cryptography to the idealized-cryptography game. The details of this mapping are not relevant to our treatment and the actual definition and its connection to the simulation paradigm has already been established in [HPS16]. However, for readers who might be questioning whether the notion of pseudo-Nash is an important step in the direction of devising tools for non-crypto-experts to reason about games involving cryptography we find it useful to contrast our notion to the above definition (Definition B.3) which we have included in the appendix.

In addition to the evident difference in complexity of the definitions, there are several qualitative and quantitative arguments in favor of pseudo-Nash equilibrium (PNE) compared to computational Nash equilibrium (CNE): First, as discussed above, CNE can only be defined for a very restricted class of games; and even for this class, to make the analysis tractable one would typically require that the game itself is derived from an idealized-cryptography game, and even then, the analysis would need some degree of familiarity with advanced cryptographic reasoning (e.g., the simulation paradigm). In contrast, PNE is defined for any game. And for the special class of games that use cryptography, we equip it with an automatic translation theorem that ensure one can replace idealized cryptography with real-world cryptography. Second, existence of a Nash equilibrium does not necessarily imply existence of CNE in contrast to PNE (see both Section 6.1 and Appendix B.1). Third, CNE is sensitive to the magnitude of the utilities of different events. For example, events which occur with negligible probability but incur huge (exponential) utility make the reasoning from [HPS16] inapplicable and, as we demonstrate, eliminate the possibility of CNE. This is in contrast to PNE which, by definition, renders negligible probability events irrelevant. As we show in Section 6.1, this distinction makes a big difference in the analysis of classical rational-cryptography games, e.g., rational secret sharing [HT04]. We note that one might think that this third distinction is somewhat esoteric, as in traditional game theory it is reasonable to assume that utilities are constant. This is however not the case: first, when dealing with parameterized games that use asymptotic cryptography it is reasonable to assume that the utilities depend on the underlying parameter. And if this is the case, it is unclear why one should be limiting this dependency to be polynomial. In fact, one can think of very natural games (even without using cryptography) where such utility would be exponential in a specific parameter (e.g., the size of the player's strategies): Imagine a version of the guessing game (Example 1.2) where the leader (Player 1) chooses an κ -bit string but sends nothing to the follower (Player 2); then the follower announces a string (an attempt to guess the leader's choice); every bit he guesses correctly doubles his reward. It is very simple to prove that the unique Nash in this game is for both players to chose uniformly random strings. However, by a similar reasoning as in the original guessing game one can show that the above strategy cannot be proven to be CNE (and the aforementioned blueprint to prove CNE from [HPS16] does not help here, since there is no idealization of cryptography happening here). In contrast, since any Nash is also PNE, the above strategy is trivially also PNE. In fact, as discussed in Section 3 we view PNE as

 $^{^{2}}$ We focus here on [HPS16] but our reasoning and comparison applies to most proposed notions of equilibrium for games using cryptography, as long as they do not modify the underlying utility, e.g., by assuming that computation costs [HP15].

the natural extension of Nash to such parameterized games; furthermore, our real-to-ideal theorem, can be used to directly prove that replacing in the PNE of this game random coins with pseudo-random coins yields another PNE.

2 Preliminaries

2.1 Cryptography notation and definitions

Our goal is to develop tools that allow a game-theorist with little to no exposure to cryptography to reason about games that use cryptography. Our results offer a seamless translation from games with idealized cryptography to games with real cryptography. In this section, we outline the basic definitions and tools from the cryptographic literature that are needed to understand the translation.

State-of-the-world assumptions that are typical in cryptography rely on asymmetries that (are assumed to) exist in certain problems. E.g., one-way functions are, intuitively, easy to compute and hard to invert, pseudo-random generators are efficiently computable length-expanding deterministic algorithms but their output on random inputs is hard to distinguish from samples from a random distribution, etc. However, "hardness" of a problem does not make it impossible to solve; in fact, most cryptographic hardness assumptions only apply to hardness on average (i.e., on random inputs). As such, a cryptographic statement would typically embrace the idea that the adversary might with "tiny" probability violate its claimed security (by violating the underlying assumption).

Capturing "tiny" in a way that allows for cryptographic proofs is tricky. For example, in an encryption scheme, we would want that the adversary has a tiny probability of recovering information about the plaintext. But if the key is small, say 5 bits, then with probability $2^{-5} = 1/32$ the adversary can guess it and recover the whole plaintext. As such, the notion of tiny is defined to correspond to "eventually tiny," i.e., for cryptographic keys whose size is larger than some value κ_0 , usually referred to as the *security parameter*. This gives rise to the following notion of *negligibility* aimed at capturing the above intuition of eventually tiny:

Definition 2.1. A function $\delta : \mathbb{N} \to \mathbb{R}$ is negligible if for every constant *c* there exists a $\kappa_c \in \mathbb{N}$ such that for all $\kappa \geq \kappa_c$ it holds $\delta(\kappa) < \frac{1}{\kappa^c}$.

Most cryptographic security definitions compare a cryptographic construction, that typically involves a key, to an ideal primitive, that the construction is supposed to securely realize. Security requires that the random variable that corresponds to the outcome of an execution of the cryptographic construction cannot be distinguished from the random variable that corresponds to the outcome of an invocation of the ideal primitive (we defer details of such a definition to Section 5). As such, the notion of (computational) indistinguishability is deeply rooted in the cryptographic method (and as we shall see, will be the key in defining a crypto-friendly notion of stability in games that use cryptography). We shortly define computational indistinguishability.

For a random variable X and a randomized algorithm \mathcal{D} with a binary output, that can draw a sample from the probability distribution P_X of X, we will denote by $\mathcal{D}^X(x)$ the random variable which corresponds to the outcome of \mathcal{D} on input an x sampled from X. We will also denote by 1^{κ} the unary representation of κ , i.e., the string consisting of κ ones.

Definition 2.2. [Computational Indistinguishablity] Let $X = \{X_{\kappa}\}_{\kappa \in \mathbb{N}}, Y = \{Y_{\kappa}\}_{\kappa \in \mathbb{N}}$ be a pair of ensembles of random variables. X is computational indistinguishable from Y, denoted by $X \cong Y$, if there exists a negligible function $negl(\cdot) : \mathbb{N} \to [0, 1]$ such that for every probabilistic polynomial time (PPT) algorithm \mathcal{D} with a binary output, often called the distinguisher, the following holds:

$$\left|\Pr\left[\mathcal{D}^{X_{\kappa}}(1^{\kappa})=1\right]-\Pr\left[\mathcal{D}^{Y_{\kappa}}(1^{\kappa})=1\right]\right|\leq \operatorname{\textit{negl}}(\kappa).$$

The quantity $|\Pr[\mathcal{D}^{X_{\kappa}}(1^{\kappa})=1] - \Pr[\mathcal{D}^{Y_{\kappa}}(1^{\kappa})=1]|$ is often referred to as the distinguishing advantage of \mathcal{D} in distinguishing X_{κ} from Y_{κ} , as it captures, intuitively, how similar X_{κ} and Y_{κ} look to \mathcal{D} . In fact, it is not hard to see that if we do not restrict \mathcal{D} to be PPT, i.e., if we take the distinguishing advantage of the best (even inefficient) distinguisher, then this will be equal to the standard statistical distance between the X_{κ} and Y_{κ} . For the curious reader, we note in passing that providing \mathcal{D} the input 1^{κ} is to ensure that it runs in time polynomial in κ , which is needed to avoid trivial counter-examples of rendering cryptographic constructions secure because the distinguisher does not have sufficient time to parse X_{κ} (see [Gol03] for a detailed discussion).

2.2 Game theory notations and definitions

We define the notion of Pseudo-Nash on standard normal form games. We denote by $\mathcal{G}([n], \mathcal{N}, U)$ a game with n players and with action set $\mathcal{N} = \mathcal{N}_1 \times \ldots \times \mathcal{N}_n$. For a mixed strategy profile $s \in \mathcal{N}$ we are interested in the utility random variable $U_i(s_i, s_{-i})$ of player $i \in [n]$. In such games, Nash equilibrium is defined as usually:

Definition 2.3 (Nash Equilibrium). A strategy profile $s = (s_1, \ldots, s_n)$ is a Nash equilibrium (NE) of a game \mathcal{G} if, for all $i \in [n]$ and for all deviating strategies \hat{s}_i it holds $\mathbb{E}[U_i(s)] \ge \mathbb{E}[U_i(\hat{s}_i, s_{-i})]$.

Note that, as we show, the notion of Pseudo-Nash can be extended to parameterized games—like the guessing game from the introduction, modified so that player one has a variable size input and the utilities depend on this size.

3 Empirical Dominance and Pseudo-Nash Equilibrium

We introduce a new notion of ordering random variables with respect to their empirical means. We say that a random variable (m, δ) -empirically X means dominates Y (EM dominates for short) if the empirical mean of X tends to be higher than Y. Formally,

Definition 3.1 (Empirical Mean Dominance). Let X, Y be a pair of random variables. Then X (m, δ) -empirical mean dominates Y, denoted by $X \underset{m,\delta}{\geq} Y$, for $m \in \mathbb{N}^+$ and $\delta \in [0, 1]$:

$$\Pr\left[\frac{1}{m}\sum_{j=1}^{m}Y^{(j)} \ge \frac{1}{m}\sum_{j=1}^{m}X^{(j)}\right] - \Pr\left[\frac{1}{m}\sum_{j=1}^{m}X^{(j)} \ge \frac{1}{m}\sum_{j=1}^{m}Y^{(j)}\right] \le \delta$$

where $X^{(j)}, Y^{(j)}$ are *i.i.d.* samples from X, Y respectively.

As the name suggests, our goal is to use empirical dominance to compare (mixed) strategies, s and \hat{s} , in normalform games. In this context, the random variables X and Y will indicate the utilities of s and \hat{s} , respectively in the game.³ In order to account for computational considerations and enable cryptographic reasoning, we make the following restriction on empirical dominance.

Definition 3.2. [Computational Mean Dominance] Let $X = \{X_{\kappa}\}_{\kappa \in \mathbb{N}}$, $Y = \{Y_{\kappa}\}_{\kappa \in \mathbb{N}}$ be a pair of random variable ensembles (i.e., infinite sequences parameterized by κ). Then X computational mean dominates Y, denoted by $X \gtrsim Y$, if for all constants $c \ge 1$, there exist a constant $\hat{c} > c$ and a $\kappa_0 > 0$ such that for all $\kappa \ge \kappa_0$:

$$\Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y_{\kappa}^{(j)} \ge \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)} \ge \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y_{\kappa}^{(j)}\right] < \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)} \ge \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)} \le \frac{1}{\kappa^{\hat{c}}}X_{\kappa}^{(j)} \le \frac{1}{\kappa^{\hat{c}}}X_{\kappa}^{(j)}$$

where $X_{\kappa}^{(j)}, Y_{\kappa}^{(j)}$ are *i.i.d.* samples from X_{κ}, Y_{κ} respectively.

As discussed in the introduction, the intuitive interpretation of the above $X \gtrsim Y$ dominance definition is as follows. Consider a distinguisher who uses the empirical means of X and Y as a statistical test to decide whether or not the expected value of X is above the expected value of Y. As long as this distinguisher takes at most polynomially many—i.e., $\kappa^{\hat{c}}$ for some constant \hat{c} —samples of X and Y, with all but negligible —eventually diminishing faster than $1/k^c$ for any c—probability, he will decide that X dominates Y.

In the context of analyzing games, the parameter κ in the above ensembles corresponds to the size of the game's description, which is a natural input to any distinguisher performing the above statistical test. As such the above notion applies to comparing two strategies s and \hat{s} in standard (fixed) games similarly to Definition 3.1, by setting $X_{\kappa} = X_1$ and $Y_{\kappa} = Y_1$, for all $k \in \mathbb{N}$, where X_1 and Y_1 are the utilities of the two strategy profiles, s and \hat{s} , respectively. In fact, somewhat surprising, as we prove in Lemmas 3.1 and 3.2, restricting the notion of empirical

 $^{^{3}}$ Note that we mean the actual utility that players receive when they play these strategies rather than expected utility.

dominance as above yields a definition equivalent to standard dominance of expectations, something which is not true otherwise. Looking ahead this will render our Pseudo-Nash notion equivalent to Nash in such games, which in our opinion is the ultimate "sanity check" for any such notion.

Most interestingly for the goals of our paper, by incorporating ensembles into the definition we are now able to reason about parameterized games—think of the guessing game from our introduction where the size of the input string is a parameter and affects the utility (e.g., the more bits of player 1's input that player 2 guesses the higher his utility). We believe that pseudo-Nash is the natural adaptation of classical Nash to such games. Most importantly, it means that we can now seemlesly capture games using cryptography by simply making the security parameter (in unary notation, 1^{κ}) part of the game description.

Given the above notion of dominance, we can now define our notion of pseudo-Nash equilibrium. Similar to Nash equilibrium, a strategy profile is a pseudo-equilibrium if for all players, the random variable (ensembles) that corresponds to their utility dominates all other utility random variables that come from a unilateral deviation. Formally, we define pseudo-equilibrium.

Definition 3.3 (Pseudo-equilibrium). A computational Pseudo equilibrium of a (possibly parameterized) game $\mathcal{G}([n], \mathcal{N}, U)$ is a strategy profile $s = (s_1, \dots, s_n) \in \mathcal{N}$ that for all $i \in [n]$ and for all $\hat{s}_i \in \mathcal{N}_i$, the utility random variable (ensemble) of player i according to $s_i, U_i(s_i, s_{-i}) = \{U_i^{\kappa}(s)\}_{\kappa \in \mathbb{N}}$, computationally EM dominates the utility variable (ensemble) of player i according to $\hat{s}_i, U_i(\hat{s}_i, s_{-i}) = \{U_i^{\kappa}(s)\}_{\kappa \in \mathbb{N}}$. That is, s is a computational pseudo equilibrium if $\forall i \in [n], \forall \hat{s}_i \in \mathcal{N}_i : U_i(s_i, s_{-i}) \gtrsim U_i(\hat{s}_i, s_{-i})$.

3.1 Equivalence to Nash for Standard Games

In this section, we show that for standard fixed description (i.e., non-parametrized) games, our pseudo-equilibrium notion is equivalent to Nash equilibrium.

Theorem 3.1 (Equivalence of Nash and Pseudo-Nash Equilibria). For a normal-form game $\mathcal{G}([n], \mathcal{N}, U)$, a strategy profile $s = (s_1, \ldots, s_n)$ is a Nash equilibrium if, and only if, it is a pseudo-Nash equilibrium according to definition Definition 3.3.

Proof. To prove this statement, we need the following three technical lemmas.

Lemma 3.1. Let X, Y be two random variables with bounded support R such that $\mathbb{E}[X] > \mathbb{E}[Y]$. Then X computationally dominates Y, $X \gtrsim Y$, according to definition Definition 3.2.

Proof. Let us define $\Delta = \mathbb{E}[X] - \mathbb{E}[Y] > 0$. By Definition 3.2, we want to show that $X \gtrsim Y$. That is, for all constants $c \geq 1$, there exist a $\hat{c} > c$, which we set to $\hat{c} = 4c$, and a $\kappa_0 > 0$ such that for all $\kappa \geq \kappa_0$:

$$\Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}Y_{\kappa}^{(j)} \ge \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}X_{\kappa}^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}X_{\kappa}^{(j)} \ge \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}Y_{\kappa}^{(j)}\right] < \frac{1}{\kappa^{c}}$$
(1)

where $X_{\kappa}^{(j)}, Y_{\kappa}^{(j)}$ are i.i.d. samples from X_{κ}, Y_{κ} respectively.

Since X and Y are random variables Equation (1) is reduced to:

$$\Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}Y^{(j)} \ge \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}X^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}X^{(j)} \ge \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}Y^{(j)}\right] < \frac{1}{\kappa^{c}}$$
(2)

For a fixed κ and constant c, define the empirical averages: By Hoeffding's inequality [Hoe63], for any $\epsilon > 0$ we have

$$\Pr\left[\left|\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}X^{(j)} - \mathbb{E}\left[X\right]\right| \ge \epsilon\right] = \Pr\left[\left|\sum_{j=1}^{\kappa^{4c}}X^{(j)} - \kappa^{4c}\mathbb{E}\left[X\right]\right| \ge \kappa^{4c}\epsilon\right] \le 2\exp\left(-\frac{\kappa^{4c}\epsilon^2}{R^2}\right)$$

and

$$\Pr\left[\left|\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}Y^{(j)} - \mathbb{E}\left[Y\right]\right| \ge \epsilon\right] = \Pr\left[\left|\sum_{j=1}^{\kappa^{4c}}Y^{(j)} - \kappa^{4c}\mathbb{E}\left[Y\right]\right| \ge \kappa^{4c}\epsilon\right] \le 2\exp\left(-\frac{\kappa^{4c}\epsilon^2}{R^2}\right)$$

For ease of notation, let us define for fixed but arbitrary κ, c :

$$\overline{X} = \frac{1}{\kappa^{4c}} \sum_{j=1}^{\kappa^{4c}} X^{(j)} \quad \text{and} \quad \overline{Y} = \frac{1}{\kappa^{4c}} \sum_{j=1}^{\kappa^{4c}} Y^{(j)}$$

Set $\epsilon = \Delta/4$. We consider the following two events, $\overline{X} \ge \mathbb{E}[X] - \Delta/4$ and $\overline{Y} \le \mathbb{E}[Y] + \Delta/4$. When both events occur we have that $\overline{X} - \overline{Y} \ge \mathbb{E}[X] - \Delta/4 - (\mathbb{E}[Y] + \Delta/4) \ge \Delta/2 > 0$

$$\begin{aligned} \Pr\left[\overline{X} > \overline{Y}\right] &\geq \Pr\left[\left(\overline{X} \ge \mathbb{E}\left[X\right] - \Delta/4\right) \& \left(\overline{Y} \le \mathbb{E}\left[Y\right] + \Delta/4\right)\right] \\ &= \Pr\left[\left(\overline{X} \ge \mathbb{E}\left[X\right] - \Delta/4\right)\right] \cdot \Pr\left[\left(\overline{Y} \le \mathbb{E}\left[Y\right] + \Delta/4\right)\right] \\ &\geq \left(1 - 2\exp\left(-\frac{\kappa^{4c}(\Delta/4)^2}{R^2}\right)\right) \left(1 - 2\exp\left(-\frac{\kappa^{4c}(\Delta/4)^2}{R^2}\right)\right) \\ &\geq 1 - 4\exp\left(-\frac{\kappa^{4c}\Delta^2}{8R^2}\right) \end{aligned}$$

To conclude the proof, we need to show that there exists a κ_0 such that for all $\kappa \geq \kappa_0$ and for all $c \geq 1$ Equation (2) holds. We can see that it suffices to find the number of samples required to obtain $\Pr\left[\overline{X} > \overline{Y}\right] \geq \frac{1}{2}$. That is because $\Pr\left[\overline{Y} > \overline{X}\right] - \Pr\left[\overline{X} > \overline{Y}\right] \leq 0 \leq \frac{1}{\kappa^c}$. Thus, we need

$$4 \exp\left(-\frac{\kappa^{4c}\Delta^2}{8R^2}\right) \le \frac{1}{2} \Leftrightarrow -\frac{\kappa^{4c}\Delta^2}{8R^2} + \ln 4 \le -\ln 2 \Leftrightarrow \kappa^{4c} \ge \frac{8R^2}{\Delta^2} \ln 8$$

t suffices to set $\kappa_0 = \left(\frac{8R^2}{\Delta^2} \ln 8\right)^{1/4}$.

Lemma 3.2. Let X, Y be two random variables with bounded support R such that $\mathbb{E}[X] = \mathbb{E}[Y]$. Then X computationally dominates $Y, X \gtrsim Y$, and Y computationally dominates $X, Y \gtrsim X$, according to definition Definition 3.2.

Proof. For two random variables X, Y with $\mathbb{E}[X] = \mathbb{E}[Y]$, we want to show that $X \gtrsim Y$ and $Y \gtrsim X$. Since $X \gtrsim Y$ and $Y \gtrsim X$ are symmetric, without loss of generality we will show that $X \gtrsim Y$. That is, for all constants $c \ge 1$, there exist $\hat{c} > c$ and a $\kappa_0 > 0$ such that for all $\kappa \ge \kappa_0$:

$$\Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y^{(j)} \ge \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X^{(j)} \ge \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y^{(j)}\right] < \frac{1}{\kappa^{c}}$$
(3)

where $X^{(j)}, Y^{(j)}$ are i.i.d. samples from X, Y respectively.

Thus, since $c \ge 1$ if

We will consider the random variable Z = X - Y. By linearity of expectations we have that $\mathbb{E}[Z] = \mathbb{E}[X] - \mathbb{E}[Y] = 0$. For some fixed but arbitrary κ , \hat{c} let the empirical mean of Z with $\kappa^{\hat{c}}$ i.i.d samples be

$$\overline{Z}_{\kappa,\hat{c}} = \frac{1}{\kappa^{\hat{c}}} \sum_{j=1}^{\kappa^{\hat{c}}} Z^{(j)} = \frac{1}{\kappa^{\hat{c}}} \sum_{j=1}^{\kappa^{\hat{c}}} \left(X^{(j)} - Y^{(j)} \right).$$

Since the random variable Z has finite support, we know that its moments are also finite. Let $\sigma^2 = \mathbb{E}\left[Z^2\right] < \infty$ and $\rho = \mathbb{E}\left[|Z|^3\right] < \infty$. By the Berry–Esseen theorem [Ber41, Ess42], considering the cumulative distribution function (CDF) $F_{\kappa,c}(\cdot)$ of $\frac{\overline{X}_{\kappa,\hat{c}}\sqrt{\kappa^{\hat{c}}}}{\sigma}$ and the CDF of normal distribution $\Phi(\cdot)$ we have that for a C < 0.47:

$$\sup_{x \in \mathbb{R}} |F_{\kappa,c}(x) - \Phi(x)| \le \frac{C\rho}{\sigma^3 \sqrt{\kappa^{\hat{c}}}}$$
(4)

To satisfy Equation (3) we need to show that for all constants c and a $\hat{c} > c$ there exists κ_0 such that for all $\kappa \ge \kappa_0$ we have $\Pr\left[\overline{Z}_{\kappa,\hat{c}} < 0\right] - \Pr\left[\overline{Z}_{\kappa,\hat{c}} > 0\right] < \frac{1}{\kappa^c}$. Notice that we can remove the event that $\overline{Z}_{\kappa,\hat{c}} = 0$ from both probabilities without changing their difference. From Equation (4) we have that $\Pr\left[\overline{Z}_{\kappa,c} < 0\right] \le \Pr\left[\overline{Z}_{\kappa,c} \le 0\right] \le \frac{C\rho}{\sigma^3\sqrt{\kappa^c}} + \Phi(0) = \frac{C\rho}{\sigma^3\sqrt{\kappa^c}} + \frac{1}{2}$. We can easily show that

$$\Pr\left[\overline{Z}_{\kappa,\hat{c}} > 0\right] = 1 - \Pr\left[\overline{X}_{\kappa,\hat{c}} \le 0\right] \ge 1 - \left(\frac{1}{2} + \frac{C\rho}{\sigma^3 \kappa^{2c}}\right) = \frac{1}{2} - \frac{C\rho}{\sigma^3 \kappa^{2c}}$$

Plugging these inequalities in to Equation (3) we get:

$$\Pr\left[\overline{Z}_{\kappa,\hat{c}} < 0\right] - \Pr\left[\overline{Z}_{\kappa,\hat{c}} > 0\right] \le \frac{1}{2} + \frac{C\rho}{\sigma^3\sqrt{\kappa^{\hat{c}}}} - \left(\frac{1}{2} - \frac{C\rho}{\sigma^3\sqrt{\kappa^{\hat{c}}}}\right) \le \frac{2C\rho}{\sigma^3\sqrt{\kappa^{\hat{c}}}}$$

Therefore, we want that for all $c \ge 1$

$$\frac{2C\rho}{\sigma^3\sqrt{\kappa^{\hat{c}}}} < \frac{1}{\kappa^c} \implies \kappa^{\frac{\hat{c}}{2}-c} > \frac{2C\rho}{\sigma^3}$$

By selecting $\hat{c} = 2c + 2$, we get that

$$\kappa > \frac{2C\rho}{\sigma^3}$$

That concludes the proof.

Lemma 3.3. Let X, Y be two random variables with bounded support R. If X computationally dominated Y, according to definition Definition 3.2, then $\mathbb{E}[X] \ge \mathbb{E}[Y]$.

Proof. We are going to prove this lemma by contradiction. By the fact that $X \gtrsim Y$ we have that for all constants $c \geq 1$, there exist $\hat{c} > c$ and a $\kappa_0 > 0$ such that for all $\kappa \geq \kappa_0$:

$$\Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y^{(j)} \ge \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X^{(j)} \ge \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y^{(j)}\right] < \frac{1}{\kappa^{c}}$$
(5)

where $X^{(j)}, Y^{(j)}$ are i.i.d. samples from X, Y respectively.

For the rest of the proof, it suffices to set $\hat{c} = 4c$. For ease of notation, let us define for fixed but arbitrary κ, c the empirical mean random variable of X and Y:

$$\overline{X} = \frac{1}{\kappa^{4c}} \sum_{j=1}^{\kappa^{4c}} X^{(j)} \quad \text{and} \quad \overline{Y} = \frac{1}{\kappa^{4c}} \sum_{j=1}^{\kappa^{4c}} Y^{(j)}$$

Assume that $\mathbb{E}[Y] > \mathbb{E}[X]$. Let $\Delta = \mathbb{E}[Y] - \mathbb{E}[X] > 0$ We will show that for all $c \ge 1$ there exists a κ_1 such that for all $\kappa \ge \kappa_1$ Equation (5) does not hold. Notice that Lemma 3.1 does not directly contradict Equation (5) since the lemma implies $\Pr[\overline{Y} > \overline{X}] - \Pr[\overline{X} > \overline{Y}] \ge \frac{1}{\kappa^c}$ Hence, we $\Pr[U_{\hat{s}} > U_s] - \Pr[U_s > U_{\hat{s}}] \ge -1/\kappa^c$. We need to show a tighter bound.

Towards the contradiction we will show that $\Pr\left[\overline{Y} > \overline{X}\right] - \Pr\left[\overline{X} > \overline{Y}\right] \ge \frac{2}{\kappa^c} \ge \frac{1}{\kappa^c}$. Following the same steps as in Lemma 3.1.

By Hoeffding's inequality, for any $\epsilon > 0$ we have:

$$\Pr\left[\left|\overline{X} - \mathbb{E}\left[X\right]\right| \ge \epsilon\right] \le 2\exp\left(-\frac{2\kappa^{4c}\epsilon^2}{R^2}\right)$$

and

$$\Pr\left[\left|\overline{Y} - \mathbb{E}\left[\overline{Y}\right]\right| \ge \epsilon\right] \le 2\exp\left(-\frac{2\kappa^{4c}\epsilon^2}{R^2}\right)$$

Setting $\epsilon = \Delta/4$, we have that,

$$\begin{aligned} \Pr\left[\overline{Y} > \overline{X}\right] &\geq \Pr\left[\overline{Y} \geq \mathbb{E}\left[Y\right] - \Delta/4\right] \cdot \Pr\left[\overline{X} \leq \mathbb{E}\left[X\right] + \Delta/4\right] \\ &\geq \left(1 - 2\exp\left(-\frac{\kappa^{4c}\Delta^2}{8R^2}\right)\right)^2 \geq 1 - 4\exp\left(-\frac{\kappa^{4c}\Delta^2}{8R^2}\right) \end{aligned}$$

To obtain a tighter to contradict Equation (5) bound we need to find a κ_0 such that for all $\kappa \geq \kappa_0$ and for all c the $\Pr[\overline{Y} > \overline{X}] \geq 1/2 + 1/\kappa^c$. That implies $\Pr[\overline{X} > \overline{Y}] \leq 1/2 - 1/\kappa^c$. Combining the two we get that

 $\Pr\left[\overline{X} > \overline{Y}\right] - \Pr\left[\overline{Y} > \overline{X}\right] \leq -\frac{2}{\kappa^c}$ which directly contradicts Equation (5). We can find the number of samples required by

$$\Pr\left[\overline{Y} > \overline{X}\right] \ge 1 - 4\exp\left(-\frac{\kappa^{4c}\Delta^2}{8R^2}\right) \ge 1/2 + \frac{1}{\kappa^c} \Leftrightarrow 4\exp\left(-\frac{\kappa^{4c}\Delta^2}{8R^2}\right) \le \frac{1}{2} - \frac{1}{\kappa^c}$$

We can easily see that $4 \exp\left(-\frac{\kappa^{4c}\Delta^2}{8R^2}\right)$ is an decreasing function of κ that goes to 0 as κ goes to infinity. On the other hand $\frac{1}{2} - \frac{1}{\kappa^c}$ is an increasing function of κ that goes to 1/2 as κ goes to infinity. Furthermore because of the monotonicity, if the inequality holds for some constant c it also holds for all c' > c. Thus, it suffices to show it for c = 1. Finally, to show there exists a κ_0 , consider any constant $\epsilon \in (1/2)$ such that $4 \exp\left(-\frac{\kappa^4\Delta^2}{8R^2}\right) < \epsilon$ and $\epsilon < 1/2 - 1/\kappa$. Since we want both to be satisfied simultaneously we get

$$\kappa_0 > \max\left\{\frac{8R^2}{\Delta^2}\ln\left(\frac{4}{\epsilon}\right), \frac{2}{1-2\epsilon}\right\}$$

That concludes the proof of Lemma 3.3.

Given the above three lemmas, the proof of Theorem 3.1 proceeds as follows: We will show each direction separately.

 (\implies) Nash implies Pseudo. We want to show that a Nash equilibrium also satisfies Definition 3.3. That is, there exist an κ_0 such that for all $\kappa \geq \kappa_0$ and for all players *i* and for all alternative strategies \hat{s}_i :

$$\Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i^{(j)}(\hat{s}_i;s_{-i}) > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i^{(j)}(s)\right] - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i^{(j)}(s) > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i^{(j)}(\hat{s}_i;s_{-i})\right] \le \frac{1}{\kappa^c}$$
(6)

Since the strategy profile $s = (s_1, \ldots, s_n)$ is a Nash equilibrium, we have for all $i \in [n]$ and for every unilateral deviation $\hat{s}_i \in \mathcal{N}_i$ of player i the $\mathbb{E}[U_i(s_i; s_{-i})] \geq \mathbb{E}[U_i(\hat{s}_i; s_{-i})]$. Let us define $\Delta = \min_{i,\hat{s}_i} \{\mathbb{E}[U_i(s_i; s_{-i})] - \mathbb{E}[U_i(\hat{s}_i; s_{-i})]\} \geq 0$. By construction of the game, the utilities are bounded, i.e., for all strategy profiles $U_i(\cdot) \in [a, b]$. Let R = |b - a| be the range. Consider the play i and the deviating strategy \hat{s}_i that minimizes the difference in expectation Δ .

Case 1: $\Delta > 0$. In this case, we can directly apply Lemma 3.1, to show that $U_i(s_i; s_{-i}) \gtrsim U_i(\hat{s}; s_{-i})$.

Case 2: $\Delta = 0$ In this case we can directly apply Lemma 3.2

(\Leftarrow) **Pseudo implies Nash.** Assume that the strategy profile $s = (s_1, \ldots, s_n)$ is a pseudo-Nash equilibrium. By definition, for every player $i \in [n]$, for every unilateral deviation $\hat{s}_i \in \mathcal{N}_i$, for every constant *c*there exists a sample size κ_0 such that for all $\kappa \geq \kappa_0$ the following holds:

$$\Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i^{(j)}(\hat{s}_i;s_{-i}) > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i^{(j)}(s)\right] - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i^{(j)}(s) > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i^{(j)}(\hat{s}_i;s_{-i})\right] \le \frac{1}{\kappa^c}$$
(7)

where $U_i^{(j)}(s_i, s_{-i})$ and $U_i^{(j)}(\hat{s}_i, s_{-i})$ denotes j^{th} independent sample from the corresponding utility random variables. By Lemma 3.3, this condition implies that for every $i \in [n]$ and every deviation $\hat{s}_i \in \mathcal{N}_i$,

$$\mathbb{E}\left[U_i(s_i; s_{-i})\right] \ge \mathbb{E}\left[U_i(\hat{s}_i; s_{-i})\right]$$

which is exactly the Nash equilibrium condition.

3.2 Beyond Nash for Parameterized Games

Recall that our goal is to define a notion of stability for games using cryptography, which eliminates unnatural Nash equilibria—e.g., bruteforceing crypto. The above equivalence theorem makes one wonder, if Nash is equivalent to pseudo-Nash, how is this possible. Our next sanity check is an example showing that in (random variable ensembles corresponding to) parameterized games the above equivalence between Nash and Pseudo-Nash does no longer hold. To this direction

we describe two random variable ensembles $X = \{X_{\kappa}\}_{\kappa \in \mathbb{N}}, Y = \{Y_{\kappa}\}_{\kappa \in \mathbb{N}}$ for which $\mathbb{E}[Y_{\kappa}] > \mathbb{E}[X_{\kappa}]$ for all κ ; but, the empirical mean of Y_{κ} does not converge to its expectation with polynomial samples in κ , which would make the two equilibrium notions distinct.

Example 3.1. Consider the following two random variable ensembles $X = \{X_{\kappa}\}_{\kappa \in \mathbb{N}}, Y = \{Y_{\kappa}\}_{\kappa \in \mathbb{N}}$. Let $X_{\kappa} = \{0 \ w.p. \ 1/2, 2 \ w.p. \ 1/2\}$ and $Y_{\kappa} = \{0 \ w.p. \ 1-1/2^{\kappa}, 2^{2\kappa} \ w.p. \ 1/2^{\kappa}\}$. It is easy to see that $\mathbb{E}[X_{\kappa}] = 1$ and $\mathbb{E}[Y_{\kappa}] = 2^{\kappa}$ hence $\mathbb{E}[X_{\kappa}] < \mathbb{E}[Y_{\kappa}]$ for all κ . However, we will show that X computationally EM dominates Y, $X \gtrsim Y$. Intuitively, the probability that $\sum_{j=1}^{\kappa^c} Y_{\kappa} > 0$ is negligible. On the other hand, the probability that $\sum_{j=1}^{\kappa^c} X_{\kappa} = 0$ is also negligible. To show that X computationally dominates Y we need to show that for all $c \ge 1$ there exist a $\hat{c} > c$ and a κ_0

such that for all $\kappa \geq \kappa_0$:

$$\Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y_{\kappa}^{(j)} > \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)} > \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y_{\kappa}^{(j)}\right] \le \frac{1}{\kappa^{c}} \tag{8}$$

Rearrangin the LHS of Equation (8)

$$\Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y_{\kappa}^{(j)} > \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)} > \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y_{\kappa}^{(j)}\right] \\ = \Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y_{\kappa}^{(j)} > \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)}\right] - \left(1 - \Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y_{\kappa}^{(j)} \ge \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)}\right]\right) \\ = 2\Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y_{\kappa}^{(j)} > \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)}\right] + \Pr\left[\frac{1}{\kappa^{c}}\sum_{j=1}^{\kappa^{c}}Y_{\kappa}^{(j)} = \frac{1}{\kappa^{c}}\sum_{j=1}^{\kappa^{c}}X_{\kappa}^{(j)}\right] - 1$$

To show that X dominates Y we will upper bound the probabilities separately. We will show that both probabilities are negligible and therefore the difference is negative, thus strictly less than $\frac{1}{\kappa^c}$. First,

$$\begin{split} \Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y_{\kappa}^{(j)} > \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)}\right] &= 1 - \Pr\left[\forall j \in [1,\kappa^{\hat{c}}]:Y_{\kappa}^{(j)} = 0\right] \\ &= 1 - \left(1 - \frac{1}{2^{\kappa}}\right)^{\kappa^{\hat{c}}} \le 1 - 1 + \frac{\kappa^{\hat{c}}}{2^{\kappa}} = \frac{\kappa^{\hat{c}}}{2^{\kappa}} \le \operatorname{negl}(\kappa) \end{split}$$

Next,

$$\begin{split} \Pr\left[\frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}Y_{\kappa}^{(j)} &= \frac{1}{\kappa^{\hat{c}}}\sum_{j=1}^{\kappa^{\hat{c}}}X_{\kappa}^{(j)}\right] &= \Pr\left[\forall j \in [1,\kappa^{\hat{c}}]:Y_{\kappa}^{(j)} = 0\right]\Pr\left[\forall j \in [1,\kappa^{\hat{c}}]:X_{\kappa}^{(j)} = 0\right]\\ &\leq \Pr\left[\forall j \in [1,\kappa^{\hat{c}}]:X_{\kappa}^{(j)} = 0\right] = \frac{1}{2^{\kappa^{\hat{c}}}} \leq \operatorname{\textit{negl}}(\kappa) \end{split}$$

That concludes the example that demonstrates a random variable dominates a random variable with much higher expected utility.

Properies of Pseudo-Nash and Crypto-Friendliness 4

The previous section demonstrated that computational mean dominance (and Pseudo-Nash) "plays well" with game theory. In this section we demonstrate that they also play well with cryptography, developing the basic tools that will allow us to prove our main theorem (stability-preserving ideal to real cryptography transition). More concretely, we will show that the definition of computational mean dominance (Definition 3.2) is compatible with computational indistinguishability (Definition 2.2). The following results show that: (i) if two random variables are computationally indistinguishable then we have bidirectional dominance (Lemma 4.1) (ii) if a random variable

X dominates a random variable Y then changing either with an indistinguishable one will not affect the dominance (Proposition 4.1 for replacing the dominating ensemble; Proposition 4.2 for replacing the dominated ensemble.)

First, we show that if two ensembles of random variables $X = \{X_{\kappa}\}_{\kappa \in \mathbb{N}}$, $Y = \{Y_{\kappa}\}_{\kappa \in \mathbb{N}}$ are computationally indistinguishable, then we have bidirectional computational empirical mean dominance.

Lemma 4.1. Let $X = \{X_{\kappa}\}_{\kappa \in \mathbb{N}}$, $Y = \{Y_{\kappa}\}_{\kappa \in \mathbb{N}}$ be a pair of random variable ensembles such that X and Y are computationally indistinguishable. Then X computationally-EM dominates Y, and Y computationally-EM dominates X. That is $X \cong Y \implies X \gtrsim Y \& Y \gtrsim X$.

Proof. We prove this lemma by contradiction. Assume that for two computationally indistinguishable random variables ensembles X and Y, the bidirectional dominance does not hold. Since $X \gtrsim Y$ and $Y \gtrsim X$ are symmetric, without loss of generality, we assume that $\neg(X \gtrsim Y)$ holds. Then we can construct a probabilistic polynomial-time (PPT) distinguisher that distinguishes X from Y with a non-negligible advantage.

By definition, $\neg(X \gtrsim Y)$ implies that there exists a constant c such that for all $\hat{c} > c$, we set $\hat{c} = 4c$, and for all κ_0 there exists a $\kappa \ge \kappa_0$ such that:

$$\Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}Y_{\kappa}^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}X_{\kappa}^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}X_{\kappa}^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}Y_{\kappa}^{(j)}\right] \ge \frac{1}{\kappa^{c}}$$
(9)

Consistently with the cryptographic literature, we assume that the distinguisher is non-uniform (i.e., can take a polynomial advice, which in our case will be the constant c guaranteed to exist from the above observation). We consider the following distinguisher $\mathcal{D}_1 : \{0,1\}^* \to \{0,1,\bot\}$ (Figure 1), that takes as input the running time token 1^{κ} and κ^{4c} i.i.d. samples from Z_{κ} , where Z_{κ} is equally likely distributed according to X_{κ} or Y_{κ} . The distinguisher then outputs (i) $\{0,1\}$ that indicates whether the samples were drawn from X_{κ} or Y_{κ} respectively (ii) \bot , which represents that \mathcal{D}_1 chooses to abort from making a guess.

Distinguisher
$$\mathcal{D}_1(1^{\kappa})$$

- 1. \mathcal{D}_1 gets c as its advice.
- 2. Compute the empirical mean of κ^{4c} i.i.d. samples from Z.

Let
$$\overline{Z} = \frac{1}{\kappa^{4c}} \sum_{i=1}^{\kappa^{4c}} Z^{(i)},$$

3. Compute the empirical mean of κ^{4c} i.i.d. samples from Y_{κ} .

Let
$$\overline{X} = \frac{1}{\kappa^{4c}} \sum_{i=1}^{\kappa^{4c}} X_{\kappa}^{(i)}$$
,
4. If $\overline{Z} > \overline{X}$, output 1; If $\overline{Z} < \overline{X}$, output 0; Otherwise output \bot .

Figure 1: Distinguisher \mathcal{D} for X, Y

First, we analyze the performance of the distinguisher \mathcal{D}_1 when the samples are drawn from X_{κ} .

$$\Pr\left[\mathcal{D}_{1}(X_{\kappa}^{(1)},\cdots,X_{\kappa}^{(\kappa^{4c})};1^{\kappa})=0\right] - \Pr\left[\mathcal{D}_{1}(X_{\kappa}^{(1)},\cdots,X_{\kappa}^{(\kappa^{4c})};1^{\kappa})=1\right]$$
$$=\Pr\left[\overline{X}_{\kappa}'<\overline{X}_{\kappa}\right] - \Pr\left[\overline{X}_{\kappa}'>\overline{X}_{\kappa}\right]$$
$$= 0.$$
(10)

Where \overline{X}'_{κ} and \overline{X}_{κ} are independent copies of the empirical mean random variable of X_{κ} with κ^{4c} samples. The last equality holds due to symmetry.

Similarly, we obtain the following when \mathcal{D}_1 has access to Y.

$$\Pr\left[\mathcal{D}_1(Y_{\kappa}^{(1)},\cdots,Y_{\kappa}^{(\kappa^{4c})};1^{\kappa})=1\right]-\Pr\left[\mathcal{D}_1(Y_{\kappa}^{(1)},\cdots,Y_{\kappa}^{(\kappa^{4c})};1^{\kappa})=0\right]$$

$$= \Pr\left[\overline{Y}_{\kappa} > \overline{X}_{\kappa}\right] - \Pr\left[\overline{X}_{\kappa} > \overline{Y}_{\kappa}\right]$$

$$= \Pr\left[\frac{1}{\kappa^{4c}} \sum_{j=1}^{\kappa^{4c}} Y_{\kappa}^{(j)} > \frac{1}{\kappa^{4c}} \sum_{j=1}^{\kappa^{4c}} X_{\kappa}^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{4c}} \sum_{j=1}^{\kappa^{4c}} X_{\kappa}^{(j)} > \frac{1}{\kappa^{4c}} \sum_{j=1}^{\kappa^{4c}} Y_{\kappa}^{(j)}\right] \qquad (Equation (9))$$

$$\geq \frac{1}{\kappa^{c}}.$$
(11)

Summing up Inequality (10) and Inequality (4), we get

$$\left(\Pr\left[\mathcal{D}_1(X_{\kappa}^{(1)},\cdots,X_{\kappa}^{(\kappa^{4c})};1^{\kappa})=0\right]-\Pr\left[\mathcal{D}_1(Y_{\kappa}^{(1)},\cdots,Y_{\kappa}^{(\kappa^{4c})};1^{\kappa})=0\right]\right)+\left(\Pr\left[\mathcal{D}_1(Y_{\kappa}^{(1)},\cdots,Y_{\kappa}^{(\kappa^{4c})};1^{\kappa})=1\right]-\Pr\left[\mathcal{D}_1(X_{\kappa}^{(1)},\cdots,X_{\kappa}^{(\kappa^{4c})};1^{\kappa})=1\right]\right)\geq\frac{1}{\kappa^c},$$

The final inequality implies that the distinguishing advantage of the distinguisher is at least $\frac{1}{2\kappa^c}$, which is non-negligible.

From the existence of the distinguisher \mathcal{D}_1 , we have that there exists a PPT distinguisher \mathcal{D}_2 that can distinguish X_{κ} from Y_{κ} with non-negligible advantage, even when given only a single sample from either X_{κ} or Y_{κ} . This result follows from a standard result of Cryptographic literature [Gol03]. The existence of \mathcal{D}_2 implies that $\neg(X \cong Y)$ and we make a contradiction.

Proposition 4.1. Let $X = \{X_{\kappa}\}_{\kappa \in \mathbb{N}}, \hat{X} = \{\hat{X}_{\kappa}\}_{\kappa \in \mathbb{N}}, Y = \{Y_{\kappa}\}_{\kappa \in \mathbb{N}}$ be random variable ensembles, such that X, \hat{X} are computationally indistinguishable, $X \cong \hat{X}$. Then X computationally dominates Y if, and only if, \hat{X} computationally dominates Y. That is, if $X \cong \hat{X}$ then $X \gtrsim Y \Leftrightarrow \hat{X} \gtrsim Y$.

Proof. First, we are going to show that if $X \gtrsim Y$ then $\hat{X} \gtrsim Y$. The other direction follows by repeating the proof with X and \hat{X} swapped.

We are going to prove this statement by contradiction. Assume that $\neg(\hat{X} \gtrsim Y)$, when $X \gtrsim Y$ and $X \cong \hat{X}$. Then, we will construct a PPT distinguisher \mathcal{D} that has a non-negligible advantage to distinguishing between X, \hat{X} .

By $X \gtrsim Y$ we have that for all constants $c_1 \geq 1$, there exist a constant $d_1 > c_1$ and a κ_1 such that for all $\kappa \geq \kappa_1$:

$$\Pr\left[\frac{1}{\kappa^{d_1}}\sum_{j=1}^{\kappa^{d_1}} Y_{\kappa}^{(j)} > \frac{1}{\kappa^{d_1}}\sum_{j=1}^{\kappa^{d_1}} X_{\kappa}^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{d_1}}\sum_{j=1}^{\kappa^{d_1}} X_{\kappa}^{(j)} > \frac{1}{\kappa^{d_1}}\sum_{j=1}^{\kappa^{d_1}} Y_{\kappa}^{(j)}\right] < \frac{1}{\kappa^{c_1}}$$
(12)

Similarly, by $\neg(\hat{X} \gtrsim Y)$ for we have that there exist a constant $c_2 \ge 1$ such that for all $d_2 > c_2 \kappa_2$ there exists a $\kappa \ge \kappa_2$:

$$\Pr\left[\frac{1}{\kappa^{d_2}}\sum_{j=1}^{\kappa^{d_2}} Y_{\kappa}^{(j)} > \frac{1}{\kappa^{d_2}}\sum_{j=1}^{\kappa^{d_2}} \hat{X}_{\kappa}^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{d_2}}\sum_{j=1}^{\kappa^{d_2}} \hat{X}_{\kappa}^{(j)} > \frac{1}{\kappa^{d_2}}\sum_{j=1}^{\kappa^{d_2}} Y_{\kappa}^{(j)}\right] \ge \frac{1}{\kappa^{c_2}}$$
(13)

First, we have to carefully choose the number of samples with which the distinguisher will compute the empirical mean. Notice that d_1 must be equal to d_2 since the distinguisher does not know whether the samples he receives are from X or \hat{X} . From Equation (13) we know there exists a constant c_2 that satisfies it. For Equation (12) we select a constant $c_1 > c_2$ since we know it holds for all constants. Let $d > c_1$ be a constant that satisfies Equation (12). We can see that d also satisfies Equation (13) since $d > c_1 > c_2$.

Next, we show that the PPT distinguisher \mathcal{D} in Figure 2 has a non-negligible advantage of distinguishing between X and \hat{X} . Consistently with the cryptographic literature, we assume that the distinguisher is non-uniform (i.e., can take a polynomial advice, which in our case will be the constant d guaranteed to exist from the above observation). That is, there exists a polynomial poly (\cdot) that for all κ_0 there exists a $\kappa \geq \kappa_0$:

$$\left|\Pr\left[\mathcal{D}^{X_{\kappa}}(1^{\kappa})=1\right]-\Pr\left[\mathcal{D}^{\hat{X}_{\kappa}}(1^{\kappa})=1\right]\right| \geq \frac{1}{\operatorname{poly}\left(\kappa\right)}.$$
(14)

To prove that the distinguisher has a non-negligible advantage, we need to show that for all κ_0 there exists a $\kappa \geq \kappa_0$ such that both Equation (12) and Equation (13) hold. We have already shown how we select the constants c_1, c_2, d . Next, we show that there is always a κ , for all κ_0 . It is important to note that for Equation (12) we know that there exists κ_1 that makes the inequality hold for all $\kappa \geq \kappa_1$. On the other hand, for Equation (13) we have

Figure 2: Distinguisher \mathcal{D} for X, \hat{X}

that for all κ_2 there exists a $\kappa \geq \kappa_2$ that makes the inequality hold. Since the condition for Equation (12) is more restrictive, we need to distinguish the following two cases. First, if $\kappa_0 \leq \kappa_1$, we set $\kappa_2 = \kappa_1$. From Equation (12) we have that the statement holds for all $\kappa \geq \kappa_1$, and from Equation (13) we have that there exists $\kappa \geq \kappa_2 = \kappa_1$. If $\kappa_0 > \kappa_1$, we set $\kappa_2 = \kappa_0$. Following the same logic as before, from Equation (12) we have that the statement holds for all $\kappa \geq \kappa_1$, and from Equation (13) we have that there exists $\kappa \geq \kappa_2 = \kappa_0 > \kappa_1$. Hence, we conclude that there always exists a κ that satisfies both inequalities simultaneously.

To conclude our proof, we need to lower-bound the distinguishing advantage of \mathcal{D} . First, by the construction of the distinguisher, when \mathcal{D} is given samples from X we have that,

$$\Pr\left[\mathcal{D}(X_{\kappa}^{(1)},\cdots,X_{\kappa}^{(\kappa^{d})};1^{\kappa})=1\right] - \Pr\left[\mathcal{D}(X_{\kappa}^{(1)},\cdots,X_{\kappa}^{(\kappa^{d})};1^{\kappa})=0\right]$$
$$=\Pr\left[\frac{1}{\kappa^{d}}\sum_{j=1}^{\kappa^{d}}X_{\kappa}^{(j)} > \frac{1}{\kappa^{d}}\sum_{j=1}^{\kappa^{d}}Y_{\kappa}^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{d}}\sum_{j=1}^{\kappa^{d}}Y_{\kappa}^{(j)} > \frac{1}{\kappa^{d}}\sum_{j=1}^{\kappa^{d}}X_{\kappa}^{(j)}\right]$$
$$\geq -\frac{1}{\kappa^{c_{1}}} \qquad (\text{From Equation (12)})$$

On the other hand. When the distinguisher has access to samples from \hat{X} , we have

$$\Pr\left[\mathcal{D}(\hat{X}_{\kappa}^{(1)},\cdots,\hat{X}_{\kappa}^{(\kappa^{d})};1^{\kappa})=0\right] - \Pr\left[\mathcal{D}(\hat{X}_{\kappa}^{(1)},\cdots,\hat{X}_{\kappa}^{(\kappa^{d})};1^{\kappa})=1\right]$$

$$=\Pr\left[\frac{1}{\kappa^{d}}\sum_{j=1}^{\kappa^{d}}Y_{\kappa}^{(j)} > \frac{1}{\kappa^{d}}\sum_{j=1}^{\kappa^{d}}X_{\kappa}^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{d}}\sum_{j=1}^{\kappa^{d}}X_{\kappa}^{(j)} > \frac{1}{\kappa^{d}}\sum_{j=1}^{\kappa^{d}}Y_{\kappa}^{(j)}\right]$$

$$\geq \frac{1}{\kappa^{c_{2}}} \qquad (\text{From Equation (13)})$$

Summing up Inequality (15) and Inequality (16), we have

$$\left(\Pr\left[\mathcal{D}(\hat{X}_{\kappa}^{(1)}, \cdots, \hat{X}_{\kappa}^{(\kappa^{d})}; 1^{\kappa}) = 0 \right] - \Pr\left[\mathcal{D}(X_{\kappa}^{(1)}, \cdots, X_{\kappa}^{(\kappa^{d})}; 1^{\kappa}) = 0 \right] \right)$$

+
$$\left(\Pr\left[\mathcal{D}(\hat{X}_{\kappa}^{(1)}, \cdots, \hat{X}_{\kappa}^{(\kappa^{d})}; 1^{\kappa}) = 1 \right] - \Pr\left[\mathcal{D}(X_{\kappa}^{(1)}, \cdots, \hat{X}_{\kappa}^{(\kappa^{d})}; 1^{\kappa}) = 1 \right] \right) \geq \frac{1}{\kappa^{c_{2}}} - \frac{1}{\kappa^{c_{1}}},$$

From the last inequality, we see the distinguishing advantage of \mathcal{D} is at least $\frac{1}{2}(\frac{1}{\kappa^{c_2}} - \frac{1}{\kappa^{c_1}})$. Notice that for $c_1 > c_2$ there exists a poly (\cdot) such that $\frac{1}{2}(\frac{1}{\kappa^{c_2}} - \frac{1}{\kappa^{c_1}}) \ge \text{poly}(\kappa)$. However, we know that such a distinguisher \mathcal{D} cannot exist by definition (Definition 2.2). Therefore, we conclude that $\hat{X} \gtrsim Y$.

The proof of the following proposition follows the exact same reasoning as Proposition 4.1, with only minor modifications.

Proposition 4.2. Let $X = \{X_{\kappa}\}_{\kappa \in \mathbb{N}}, Y = \{Y_{\kappa}\}_{\kappa \in \mathbb{N}}, \hat{Y} = \{\hat{Y}_{\kappa}\}_{\kappa \in \mathbb{N}}$ be random variable ensembles, such that Y, \hat{Y} are computationally indistinguishable, $Y \cong \hat{Y}$. Then X computationally dominates Y if, and only if, X computationally dominates \hat{Y} . That is, if $Y \cong \hat{Y}$ then $X \gtrsim Y \Leftrightarrow X \gtrsim \hat{Y}$.

5 Pseudo-Nash in Games that Involve Cryptography

In this section, we prove our main theorem which allows us to translate (pseudo-)Nash equilibria in games that use idealized cryptography into pseudo-Nash in a game where the idealized cryptography is replaced by a (computationally secure) cryptographic protocol.

We note that the theorem statement is aimed to be used by game theorists with little knowledge of cryptographic protocols. However, verifying the proof requires some additional tools from the relevant cryptographic theory. Therefore, we start this section with a brief overview of these tools. Readers familiar with the definitions of composable cryptography can safely skip to Section 5.1.

For clarity, we use multi-party cryptographic protocols as our running example, where parties wish to perform some joint computation in a "secure" manner. The common methodology to design and analyze such protocols can be outlined as follows:

- 1. Define the model of computation that specifies the possible actions by the parties, and the ways in which parties can interact with one another.
- 2. Specify the goal that the cryptographic protocol aims to achieve. This can be done either by describing properties that the protocol should have, e.g., for encryption, an attacker who observes the ciphertext should gain no information about the plaintext; or by specifying an idealized primitive, which in the cryptographic literature is called the ideal functionality, that embodies the goal that we are aiming to achieve. Consider for example the case of cryptographic commitments: a sender (aka the *committer*) with some input x can run an interactive protocol with a receiver (aka the *verifier*) such the following properties are satisfied:
 - (a) (hiding) At the end of the above protocol the receiver has not learned anything about x.
 - (b) (binding) The sender is "committed" to his input, i.e., there is a way to convince the receiver that x was the sender's original input, but there is no way to lie that his input was some $x' \neq x$.

The above security goals are embodied by a simple commitment functionality which upon receiving an input x from the sender records it and informs the receiver that some input was received (without leaking anything about it); and once the input has been recorded and the receiver has been informed about it, the sender can instruct the functionality to reveal it to the receiver. It is easy to verify that the interaction of a protocol with the above functionality satisfies the security properties of a commitment.

3. Specify the protocol, i.e., define the actions, and prove that it satisfies a cryptographic security definition (as discussed below).

SIMULATION-BASED SECURITY In this work, we will use the notion of *simulation-based security* as the security definition for cryptographic protocols. In addition to being the standard security definition in modern cryptographic protocols analysis, simulation-based security is also suitable for our goal of seamlessly using cryptography within game theory, due to composition theorems that it comes equipped with. In a nutshell, such theorems allow us to replace, within a bigger protocol, idealized cryptographic primitives with their cryptographic implementations, without destroying their security properties. This might sound very close to our main theorem. Therefore, it is worth here noticing a highly relevant difference between the standard approaches taken by cryptography and game theory to frame and reason about the (mis)behavior of parties in protocols and games, respectively: In particular, cryptography assumes that the adversary might corrupt several parties in a coordinated manner as opposed to individual rationality which is common in game theory. We note in passing that there are notable refinements of the cryptographic model to allow for capturing non-coordinated definitions [AKMZ12]; similarly, game-theoretic stability notions that allow coordination have been proposed (see [Hal11] for a nice survey.)

We can use any composable (simulation-based) security cryptographic framework, e.g., [Can00, Can20, BPW04] but for this exposition, we will focus on Canetti's Universal Composability (UC) framework [Can20] which is the most broadly adopted in the modern literature.

The model of computation in UC models (the strategies of the) parties as interactive Turing machines (ITMs). These are Turing machines—a standard theoretical abstraction of computing devices—with dedicated input and output tapes (corresponding to I/O interface) and communication tapes that can be connected to other ITMs' communication tapes: If two ITMs M_1 and M_2 have connected (linked) communication tapes, then M_1 can write to its communication tape shared with M_2 so that M_2 can read it and vice versa.

An interactive protocol among n parties can be described as a vector $\pi = (\pi_1, \ldots, \pi_n)$ of n ITM's, where the *i*th ITM, π_i , describes the (interactive) strategy of the *i*-th party (this can be thought of as a machine that specifies all possible reactions of an agent in an extensive form game). For simplicity, we will assume that the n parties' ITMs are not connected to one another; instead, when in the protocol we want to have Party *i* send a message to Party *j*, we will assume that they are both connected to another ITM (corresponding to a communication channel between them) that is linked to both of them.

Given the above model, a party that deviates from the protocol can be captured as a party that replaces its protocol-prescribed ITM by a different one. The actual cryptographic security definition considers situations where several parties might deviate in a coordinated (by an adversary) manner. However, for the purpose of this work—since we only consider Nash equilibrium with individually rational players—it suffices to discuss the definition restricted to any one of the parties being corrupted (and deviating).

The security definition of a cryptographic protocol is given as follows: First, we specify the ideal goal by means of an ideal functionality \mathcal{F} (also an ITM), which as in Step 1, above, embodies the idealized version of our protocols goals, e.g., in the example above, a commitment functionality, which is an ITM with the behavior discussed above, that has communication tapes linked with the committer and the verifier. Then the definition of what it means for a protocol Π —which, recall, is a profile of ITMs—to securely realize \mathcal{F} against with a single corruption is devised by comparing an execution of the protocol (the so-called real world) to an ideal invocation of the functionality \mathcal{F} (the so-called ideal world). In a nutshell, we require that there exists no distinguisher (also an ITM) that distinguish an ideal world execution from a real world execution with better than negligible probability. (In the context of UC security, a distinguisher is also called the *environment* as it defines the protocol's input/output interfaces.)

In a nutshell, a cryptographic protocol securely realizes an idealized cryptographic functionality \mathcal{F} , if any deviation in the real world can be mapped to an indistinguishable deviation in the ideal world so that the input/output behavior of the two worlds remains indistinguishable.

A bit more formally, consider a distinguisher \mathcal{D} who chooses inputs to the protocols ITMs and gets to observe their outputs. For any such \mathcal{D} , denote by $V_{\pi=(\pi_1,\ldots,\pi_n)}^{\mathcal{D},Real}$ the random variable corresponding to the view of \mathcal{D} when interacting with the protocol. (This view includes \mathcal{D} 's own input and randomness, the security parameter, and all inputs and outputs to the ITMs in π .) Similarly, $V_{(\phi_1,\ldots,\phi_n)}^{\mathcal{D},\mathcal{F},Ideal}$ corresponds to the view of \mathcal{D} in an ideal invocation of \mathcal{F} , where each ϕ_i is the "dummy" ITM that simply forwards inputs from \mathcal{D} to \mathcal{F} and hands any received inputs back to \mathcal{D} . Note that all ITMs involved in the above definition (including \mathcal{D}) are assumed to be probabilistic polynomial time (PPT).

We then say that protocol π , is secure if there exists an ITM Sim, called the *simulator*, such that for any $i \in [n]$, and any PPT ITM π'_i the view $V^{D,Real}_{(\pi'_i,\pi_{-i})}$ of D when interacting with π where party i plays π'_i instead of π_i is (computationally) indistinguishable from the view $V^{D,Ideal}_{(Sim(\pi'_i),\phi_{-i})}$ of D in an ideal invocation where party i plays the simulator's strategy (instead of the "dummy" one).

Combining the above definition—which allows to replace deviations in the cryptographic protocols with indistinguishable deviations in the idealized-crypto setting—with Proposition 4.1 already hints towards the proof of our main theorem. However, we still need to formally define games among ITMs which will allow us to use cryptography. We do so in the following:

Definition 5.1 (Normal Form Computational Game). A real normal form computational game \mathcal{G} consists of the following:

- $[n] = \{1, \dots, n\}$ is the set of players in the game. Ω is the set of all possible outcomes of the game. κ is the security parameter.
- Players' strategies are interactive, randomized Turing machines. We use $\mathcal{N}_i = \{1, 2, \cdots\}$ to index these strategies/machines. That is, player i chooses strategy $s \in \mathcal{N}_i$ means that she is playing using the sth Turing machine. This Turing machine receives a single input: the security parameter, written in unary: 1^{κ} . We write s_i^{κ} for the strategy/Turing machine chosen by player i, executed with input 1^{κ} . Let $\mathcal{N} = \mathcal{N}_1 \times \ldots \times \mathcal{N}_n$.
- The Turing machines chosen by the players can be linked via one or more Turing machines, which are fixed and part of the game description. (These will correspond to ideal functionalities, or simple communication

channels.) The security parameter κ is writte on the input tape of all ITMs in the game. All players simultaneously pick their strategies (Turing machine $s_i \in \mathcal{N}_i$ for every player i), and these Turing machines interact with each other according to the rules of the game, and a single outcome $\omega \in \Omega$ is selected (and learned by the players).

• $U_i: \Omega \mapsto \mathbb{R}$ is the random variable that indicates the utility of player *i*. We slightly overload notation and write $U_i^{\kappa}(s)$ for the utility of player *i* when the strategy profile is $s = (s_1, \ldots, s_n)$ and the security parameter is κ . We write $U_i^{\kappa}(\hat{s}_i; s_{-i})$ when player *i* unilaterally deviates to \hat{s}_i and the rest play according to s_{-i} .

5.1 Cryptographic Pseudo-Nash

Definition 5.2. We say that $\mathcal{G}^{\mathcal{F}}([n], \mathcal{N}^{\mathcal{F}}, U)$ is a computational game with access to the ideal functionality \mathcal{F} if every strategy profile $\mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_n$ in the game $\mathcal{G}^{\mathcal{F}}$ is linked to \mathcal{F} .

Let Π be a protocol that securely realizes \mathcal{F} with a single corruption. We say that $\mathcal{G}^{\Pi}([n], \mathcal{N}^{\Pi}, U)$ is the **imple**mentation of $\mathcal{G}^{\mathcal{F}}$ with Π if every strategy profile \mathcal{N}^{Π} in the game \mathcal{G}^{Π} is not linked to \mathcal{F} ; and, for every strategy profile \mathcal{N} in the game $\mathcal{G}^{\mathcal{F}}$ that interacts with \mathcal{F} , its corresponding strategy profile \mathcal{N}^{Π} in \mathcal{G}^{Π} is the one that follows exactly \mathcal{N} but executes the protocol Π whenever interaction with \mathcal{F} is required.

Theorem 5.1. Let \mathcal{F} be an ideal functionality and Π be a protocol that securely realizes \mathcal{F} against a single corruption. Let $\mathcal{G}^{\mathcal{F}}([n], \mathcal{N}^{\mathcal{F}}, U)$ be a computational game with access to the ideal functionality \mathcal{F} , and $\mathcal{G}^{\Pi}([n], \mathcal{N}^{f}, U)$ be the implementation of $\mathcal{G}^{\mathcal{F}}$ with Π . If $\sigma \in \mathcal{N}^{\mathcal{F}}$ is a computational pseudo-equilibrium (Definition 3.3) of $\mathcal{G}^{\mathcal{F}}$, then the corresponding strategy profile $s \in \mathcal{N}^{\Pi}$ of \mathcal{G}^{Π} is also a computational pseudo-equilibrium.

Proof. Let $U_i(\sigma_i, \sigma_{-i})$ be the utility random variable ensemble of player i when players playing strategy σ_i in $\mathcal{G}^{\mathcal{F}}$. Similarly, let $U_i(s_i, s_{-i}) = \{U_i^{\kappa}((s_i^*, s_{-i}^*))\}_{\kappa \in \mathbb{N}}$ be the utility random variable ensemble of player i when players playing strategy s_i in \mathcal{G}^{Π} . For ease of notation, we always use σ to denote strategy profiles in $\mathcal{G}^{\mathcal{F}}$ and s for strategy profiles in \mathcal{G}^{Π} .

We will prove the theorem by contradiction. Suppose that s is not a pseudo-equilibrium in \mathcal{G}^{Π} . By Definition 3.3, this means that there exists a player i and unilateral deviation $\hat{s}_i \in \mathcal{N}^{\Pi}$ in \mathcal{G}^{Π} such that $\neg(U_i(s_i, s_{-i}) \gtrsim U_i(\hat{s}_i, s_{-i}))$, let $\hat{s} = (\hat{s}_i, s_{-i})$. This means that strategy \hat{s}_i has observably better utility for player i. Towards the contradiction, we will show that if such a strategy exists in the real game \mathcal{G}^{Π} then there exists a strategy in the ideal game $\mathcal{G}^{\mathcal{F}}$ that also has observably better utility. By definition, this implies that σ is not a pseudo-equilibrium profile.

We first construct a strategy profile $\hat{\sigma}$ in the ideal game $\mathcal{G}^{\mathcal{F}}$ as the following: for all $j \in [n] \setminus i$, let $\hat{\sigma}_j := \sigma_j$; and let $\hat{\sigma}_i := Sim(\hat{s}_i)$, where Sim is an ITM such that $U_i(\hat{\sigma}_i, \sigma_{-i}) \cong U_i(\hat{s}_i, s_{-i})$. In other words, Sim takes as input the strategy \hat{s}_i in the real game \mathcal{G}^{Π} and outputs a strategy $(Sim(\hat{s}_i)$ in the ideal game $\mathcal{G}^{\mathcal{F}}$, such that the utility ensemble of player i when players playing $(Sim(\hat{s}_i), \sigma_{-i})$ in $\mathcal{G}^{\mathcal{F}}$ is computationally indistinguishable with the utility ensemble for player i when players playing $(\hat{s}_i, s_{-i} \text{ in } \mathcal{G}^{\Pi})$.

Next, we show that the proposed strategy profile $\hat{\sigma}$ is constructible. Equivalently, there exists an ITM Sim such that

$$U_i(Sim(\hat{s}_i), \sigma_{-i}) \cong U_i(\hat{s}_i, s_{-i}).$$

In fact, the assumption that the protocol Π securely realizes \mathcal{F} implies the existence of such a simulator Sim. Let \mathcal{D} be a distinguisher that is given a randomly chosen game (either $\mathcal{G}^{\mathcal{F}}$ or \mathcal{G}^{Π}) and observes the players' utilities after they execute their corresponding strategies. Specifically, if the game $\mathcal{G}^{\mathcal{F}}$ is played, the players follow the strategy profile (\hat{s}_i, s_{-i}) , whereas if the game \mathcal{G}^{Π} is played, they follow $(Sim(\hat{s}_i), \sigma_{-i})$. Define $V_{(\hat{s}_i, s_{-i})}^{\mathcal{D}, Real}$ as the view of the distinguisher \mathcal{D} when interacting with the real game \mathcal{G}^{Π} , and let $V_{(Sim(\hat{s}_i), \sigma_{-i})}^{\mathcal{D}, Ideal}$

Define $V_{(\hat{s}_i, s_{-i})}^{D, \text{neal}}$ as the view of the distinguisher \mathcal{D} when interacting with the real game \mathcal{G}^{Π} , and let $V_{(Sim(\hat{s}_i), \sigma_{-i})}^{D, \text{neal}}$ be the view of D when interacting with the ideal game $\mathcal{G}^{\mathcal{F}}$. Since the definition of Π securely realizing \mathcal{F} ensures the existence of such a simulator Sim, we obtain

$$V^{D,Real}_{(\hat{s}_i,s_{-i})} \cong V^{\mathcal{D},Ideal}_{(Sim(\hat{s}_i),\sigma_{-i})}$$

Noting that the view $V_{(\hat{s}_i,s_{-i})}^{\mathcal{D},Real}$ exactly equals to $U_i(Sim(\hat{s}_i),\sigma_{-i})$, and that $V_{(Sim(\hat{s}_i),\sigma_{-i})}^{\mathcal{D},Ideal}$ equals to $U_i(\hat{s}_i,s_{-i})$, we conclude that

$$U_i(\hat{\sigma}_i, \sigma_{-i}) \cong U_i(\hat{s}_i, s_{-i}).$$

Given $\neg(U_i(s_i, s_{-i}) \gtrsim U_i(\hat{s}_i, s_{-i}))$ and $U_i(\hat{\sigma}_i, \sigma_{-i}) \cong U_i(\hat{s}_i, s_{-i})$, by Proposition 4.2, we have

$$\neg (U_i(s_i, s_{-i}) \gtrsim U_i(\hat{\sigma}_i, \sigma_{-i}))$$

Moreover, since we know $U_i(\sigma_i, \sigma_{-i}) \cong U_i(s_i, s_{-i})$, by Proposition 4.1, we have

$$\neg (U_i(\sigma_i, \sigma_{-i})) \gtrsim U_i(\hat{\sigma}_i, \sigma_{-i})).$$

However, by definition, $\sigma = (\sigma_i, \sigma_{-i})$ is a computational pseudo-equilibrium of the ideal game $\mathcal{G}^{\mathcal{F}}$. This makes a contradiction.

6 Applications

In this section, we showcase our definition and our main theorem in two ways. First, we revisit one of the first results in game-theoretic cryptography concerning rational secret sharing [HT04]. We use this to demonstrate how pseudo-equilibrium eliminates counter-intuitive impossibility statements, which rely on negligible probability events. Moreover, unlike ϵ -Nash and computational Nash equilibrium [HPS16], pseudo-equilibrium remains insensitive to the magnitude of the utility associated with such events. Our second example, informally given in the introduction (Example 1.3) formally demonstrates that the Nash equilibrium of the real game is completely unreasonable, even though in the ideal game the Nash equilibrium is exactly what someone expects. However, implementing ideal cryptography with an actual cryptographic protocol, makes the ideal-game Nash a pseudo-Nash. We demonstrate this behavior with sealed bid auctions that are implemented with commitments.

6.1 HT game

The classical *t*-out-of-*n* secret sharing problem [Sha79, Bla79] involves a dealer D who wants to distribute a secret s among a group of n players, P_1, P_2, \ldots, P_n . The scheme ensures that: (1) any subset of at least t players can collaboratively reconstruct the secret without requiring further input from the dealer, while (2) any subset of fewer than t players gains no information about the secret. One of the famous examples, introduced by Shamir [Sha79], works as follows: the secret m^* is an element of a finite field \mathbb{F} , where $|\mathbb{F}| > n$. The dealer selects a random polynomial f(x) of degree at most t - 1 such that $f(0) = m^*$. Each player P_i receives a share $s_i = f(i)$. One can check that a subset of at least t players can determine f(x)—and thereby recover m^* —through polynomial interpolation. Conversely, any subset of fewer than t players learns nothing about m^* due to the random choice of the polynomial f.

In the above classical setting, there is an assumption that at least $t \ge 2n/3$ players are willing to collaborate and pool their shares when reconstructing the secret. This assumption is known to be necessary and sufficient for secret sharing with robust reconstruction against malicious (arbitrarily deviating) parties.

Later, Halpern and Teague [HT04] explore a setting where players are neither fully honest nor completely malicious but are instead *rational*, a concept known as *t-out-of-n rational secret sharing*. Depending on the utility functions of the players, one can observe that Shamir's protocol is not a Nash equilibrium in the rational setting [HT04]. Specifically, each player prefers learning the secret above all else; however, if given a choice, prefers that as few others as possible also learn the secret (an example utility is given in Table 1). Moreover, Halpern and Teague [HT04] showed that, beyond Shamir's protocol, any protocol stopping in a fixed number of rounds cannot be a Nash equilibrium.

In the model of rational secret sharing [HT04], which we refer to as the HT game in this section, there is a dealer D holding a secret m^* and n players, P_1, \ldots, P_n . A threshold t is fixed at the outset and known to all players. For our discussion, we focus on the specific case where n = 3 and t = 3. The protocol proceeds through multiple communication rounds. At the start of the HT game, D privately distributes information to each of the three players, ensuring that no subset of players gains any information about the secret m^* . After this initial distribution, the dealer no longer participates. Instead, the three players, all assumed to be rational, execute the protocol by simultaneously broadcasting messages over multiple rounds. We defer a detailed description of the HT game to Table 1, which can be found in Appendix A.1.

The Halpern-Teague protocol in the 3-out-of-3 case, denoted by s^* (a detailed description of s^* can be found in Table 2, Appendix A.1), proceeds in a sequence of *iterations*, with each iteration consisting of *four communication* rounds. During the *j*-th iteration, each player P_i flips a biased coin dc_i^j , which takes the value 1 with probability α . The players then jointly compute an *information-theoretically secure computation* to compute the value $p^j = dc_1^j \oplus dc_2^j \oplus dc_3^j$. Notably, it is impossible for any player to cheat (except by aborting the protocol) or to gain information about the values dc_i^j of other players beyond what is implied by p^j . If $p^j = dc_i^j = 1$, player P_i broadcasts their share. If all shares are revealed, the secret is reconstructed, and the game terminates. However, if $p^j = 1$ and either no shares or exactly two shares are revealed, or if the secure computation of p^j is aborted, the game ends immediately. In all other cases, the players proceed to the next iteration. Note that in the Halpern-Teague protocol s^* , the secret is reconstructed only if $dc_1^j = dc_2^j = dc_3^j = 1$ in some iteration j. Thus, assuming all players follow the protocol s^* , the protocol terminates in each iteration with probability α^3 . As a result, the number of iterations required follows a geometric distribution with parameter α^3 . This implies that the Halpern-Teague protocol s^* does not have a fixed upper bound on its round complexity. Corollary 6.1 establishes that s^* is a Nash equilibrium.

Corollary 6.1. For every $\kappa \geq 1$, the strategy $s^*(1^{\kappa})$ (shown in Table 2) is a Nash equilibrium of the HT game, shown in Table 1.

Proof. We start with a few definitions. Let $i \in \{1, 2, 3\}$ be the index of an arbitrary fixed player in the game. Let \tilde{s}_i be an arbitrary fixed strategy for player i, and let $\kappa \in \mathbb{N}$ be an arbitrary fixed integer. For brevity, we will denote $s^*(1^{\kappa})$ by s^* whenever the context of the proof is clear. Recall $s^* = (s_i^*, s_{-i}^*)$ is the strategy profile defined in the statement. Denote by $\tilde{s} = (\tilde{s}_i, s_{-i}^*)$ the unilateral deviated strategy. Denote by $U_i(s^*)$ and $U_i(\tilde{s})$ the utility random variable of player i when players playing the strategy profiles s^* and \tilde{s} , accordingly.

To show that s^* is a Nash, it suffices to show the expected utility of player *i* when players playing the strategy s^* is greater than or equal to the expected utility of player *i* when players playing \tilde{s} , or formally

$$\mathbb{E}\left[U_i(s^*)\right] \ge \mathbb{E}\left[U_i(\tilde{s})\right].$$

Note that the strategy \tilde{s}_i can only deviate by choosing to stop and hiding its secret share in some round T_0 . Let random variable \mathcal{T} denote the number of round until game stops when players playing strategy profile s^* . One can check that

$$\mathbb{E}\left[U_i(s^*) \mid \mathcal{T} \leq T_0\right] = \mathbb{E}\left[U_i(\tilde{s}) \mid \mathcal{T} \leq T_0\right].$$

On the other hand, conditioned on the event $\mathcal{T} > T_0$, we still have $\mathbb{E}[U_i(s^*) | \mathcal{T} \leq T_0] = 2^{\kappa}$ since the game when playing s^* will continue until every one learns the secret m^* . However, when playing \tilde{s} , the game will stops exactly at round T_0 . Since we have $\mathcal{T} > T_0$, and other two players follow s^* , player *i* will have only two outcomes: either only *i* learns the secret or no one learns the secret, which gives

$$\mathbb{E}\left[U_i(\tilde{s}) \mid \mathcal{T} > T_0\right] = \frac{\alpha^2}{\alpha^2 + (1-\alpha)^2} (2^{k+1} + 2) + \frac{(1-\alpha)^2}{\alpha^2 + (1-\alpha)^2}.$$

One can check that when plugging $\alpha = \frac{1}{3}$, for all $\kappa \ge 1$,

$$\mathbb{E}\left[U_i(s^*) \mid \mathcal{T} > T_0\right] \ge \mathbb{E}\left[U_i(\tilde{s}) \mid \mathcal{T} > T_0\right].$$

Combining both case, we conclude the prove by observing $\mathbb{E}[U_i(s^*)] \ge \mathbb{E}[U_i(\tilde{s})]$.

We consider a finite-round variant of the Halpern-Teague protocol, denoted as the stopping strategy \hat{s} (a detailed description of \hat{s} can be found in Table 3, Appendix A.1). \hat{s} follows exactly the strategy given by s^* but enforces termination after a fixed number of rounds, which is linear in the size of the share used in the game. Interestingly, while the Halpern-Teague protocol s^* terminates with high probability in a finite number of rounds — determined by the parameter α^3 — any strategy, e.g., \hat{s} , that enforces termination in a fixed number of rounds is ruled out as a stable equilibrium. This apparent contradiction arises because such a finite-round stopping strategy fails to satisfy the conditions of a Nash equilibrium. However, the authors prove that there exists an infinite-rounds strategy which is Nash.

What is interesting here is that the proposed Nash has an overwhelming probability of being a finite round strategy. In fact, it is not hard to find a fixed bound such that this strategy will go beyond this bound only if a negligible probability event occurs. This is exactly an artifact that pseudo-Nash was designed to take care of: As we show in Theorem 6.1, the above (finite) stopping strategy \hat{s} is *pseudo-Nash*, which resolves this counter-intuitive situation.

Theorem 6.1. The strategy \hat{s} (shown in Table 3) is a pseudo-equilibrium of the HT game, shown in Table 1.

Proof. We start with a few definitions. Let $i \in \{1, 2, 3\}$ be the index of an arbitrary fixed player in the game. Let $c \in \mathbb{N}$ be an arbitrary fixed integer. Let \tilde{s}_i be an arbitrary fixed strategy for player i. For brevity, we will denote $\hat{s}(1^{\kappa})$ by \hat{s} whenever the context of the proof is clear. Recall $\hat{s} = (\hat{s}_i, \hat{s}_{-i})$ is the strategy profile defined in the

statement. Denote by $\tilde{s} = (\tilde{s}_i, \hat{s})$ the unilateral deviated strategy. Denote by $U_i(\hat{s})$ and $U_i(\tilde{s})$ the utility random variable of player *i* when players playing the strategy profiles \hat{s} and \tilde{s} , accordingly.

To show that \hat{s} is a pseudo-equilibrium, it suffices to show there exists a κ_1 such that for all $\kappa \geq \kappa_1$:

$$\Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i(\tilde{s})^{(j)} \ge \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i(\hat{s})^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i(\hat{s})^{(j)} \ge \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_i(\tilde{s})^{(j)}\right] < \frac{1}{\kappa^c},$$

where $U_i(\hat{s})^{(j)}, U_i(\tilde{s})^{(j)}$ are i.i.d. samples from $U_i(\hat{s}), U_i(\tilde{s})$ respectively.

Let $S_{0,\kappa}$ be the event that the game, when played under the strategy profile (also referred to as the protocol) \hat{s} , ends within 4κ rounds. Following a similar argument of Corollary 6.1, one can check

 $\mathbb{E}\left[U_i(\hat{s}) \mid \mathcal{S}_{0,\kappa}\right] \geq \mathbb{E}\left[U_i(\tilde{s}) \mid \mathcal{S}_{0,\kappa}\right].$

Let Z_{κ} be the random variable denoting number of rounds until the protocol \hat{s} terminates. Let $Z_{\kappa}^{(1)}, \dots, Z_{\kappa}^{(\kappa^{4c})}$ be κ^{4c} independent copies of Z_{κ} . Let $S_{1,\kappa}$ be the event that all κ^{4c} independent runs of the game when playing the strategy profile \hat{s} ends in 4κ rounds, i.e., $Z_{\kappa}^{(j)} \leq 4\kappa$ holds for all $j \in [\kappa^{4c}]$. One can check that the event $S_{1,\kappa}$ highly likely happens. Formally, there exists a negligible function $\delta(\cdot)$ such that $\Pr[S_{1,\kappa}] \geq 1 - \delta(\kappa)$. This is because

$$\Pr \left[\mathcal{S}_{1,\kappa} \right]$$

$$= \Pr \left[Z_{\kappa} \leq 4\kappa \right]^{\kappa^{4c}} \qquad (Z_{\kappa}^{(1)}, \cdots, Z_{\kappa}^{(\kappa^{4c})} \text{ are independent})$$

$$= \left(1 - (1 - \alpha^{3})^{\kappa} \right)^{\kappa^{4c}} \qquad (\Pr \left[Z_{\kappa} \leq 4\kappa \right] = 1 - (1 - \alpha^{3})^{\kappa})$$

$$\geq 1 - \kappa^{4c} (1 - \alpha^{3})^{\kappa} \qquad (By \text{ Bernoulli's inequality})$$

$$= 1 - \frac{\kappa^{4c} 26^{\kappa}}{27^{\kappa}} \qquad (\alpha = \frac{1}{3})$$

$$= 1 - \delta(\kappa).$$

Lastly, we conclude the proof by observing that

$$\begin{split} &\Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \geq \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \geq \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)}\right] \\ &= \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)}\right] - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)}\right] \\ &\leq \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \mid \mathcal{S}_{1,\kappa}\right] \Pr\left[\mathcal{S}_{1,\kappa}\right] + \delta(\kappa) - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)}\right] \\ &\leq \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \mid \mathcal{S}_{1,\kappa}\right] + \delta(\kappa) - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)}\right] \\ &\leq \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \mid \mathcal{S}_{1,\kappa}\right] + \delta(\kappa) \\ &-\Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \mid \mathcal{S}_{1,\kappa}\right] + \delta(\kappa) \\ &-\Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \mid \mathcal{S}_{1,\kappa}\right] + 2\delta(\kappa) - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \mid \mathcal{S}_{1,\kappa}\right] \\ &\leq \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \mid \mathcal{S}_{1,\kappa}\right] + 2\delta(\kappa) - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \mid \mathcal{S}_{1,\kappa}\right] \\ &\leq \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} > \frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \mid \mathcal{S}_{1,\kappa}\right] + 2\delta(\kappa) - \Pr\left[\frac{1}{\kappa^{4c}}\sum_{j=1}^{\kappa^{4c}}U_{i}(\hat{s})^{(j)} \mid \mathcal{S}_{1,\kappa}\right] \\ &\leq 2\delta(\kappa). \qquad (\mathbb{E}\left[U_{i}(\hat{s})\mid \mathcal{S}_{0,\kappa}\right] \geq \mathbb{E}\left[U_{i}(\hat{s})\mid \mathcal{S}_{0,\kappa}\right] \right]$$

The HT game and the stopping strategy \hat{s} also well illustrate the separation between the *pseudo-equilibrium* notion and approximate equilibrium concepts, both from classical game theory, e.g., ϵ -Nash equilibrium and from past attempts to crypto-friendly definitions, e.g., *Computational Nash equilibrium* [HPS16]. Informally, a strategy profile in a computational game is a *computational Nash equilibrium* if no polynomial-time unilateral deviation results in a noticeably higher expected utility. Unlike ϵ -Nash and computational Nash equilibria, which are sensitive to the magnitude of negligible probability events (particularly in our HT game, where the utility can be exponentially large), the pseudo-equilibrium notion remains robust against such anomalies. This robustness allows us to resolve counter-intuitive impossibility results, as the stopping strategy \hat{s} fails to be an ϵ -Nash equilibrium or a computational Nash equilibrium. The formal description of these impossibility results is stated in Theorem 6.2 and Corollary 6.2.

Theorem 6.2. For every constant $\epsilon \ge 0$, and for all $\kappa > \max\{2, \lceil \frac{\ln \epsilon}{\ln 52/27} \rceil\}$, the strategy $\hat{s}(1^{\kappa})$ (shown in Table 3) is not a ϵ -Nash equilibrium of the HT game, shown in Table 1.

Proof. We start with a few definitions. Let $i \in \{1, 2, 3\}$ be the index of an arbitrary fixed player in the game. For brevity, we denote the strategy profiles $s^*(1^{\kappa})$ and $\hat{s}(1^{\kappa})$ by s^* and \hat{s} , respectively, whenever the context of the proof is clear. Denote by $U_i(s^*)$ and $U_i(\hat{s})$ the utility random variable of player i when players playing the strategy profiles s^* and \hat{s} , accordingly.

To prove the statement, it suffices to show that there exists a positive integer $\kappa_1 \in \mathbb{N}$ such that for every $\kappa \geq \kappa_1$, the following holds

$$\mathbb{E}\left[U_i(\hat{s})\right] < \mathbb{E}\left[U_i(s^*)\right] - \epsilon.$$

Let Z_{κ} be the random variable denoting number of rounds until the protocol s^* terminates. We have

$$\mathbb{E}\left[U_{i}(s^{*})\right] - \mathbb{E}\left[U_{i}(\hat{s})\right]$$

$$= \mathbb{E}\left[U_{i}(s^{*}) \mid Z_{\kappa} > 4\kappa\right] \Pr\left[Z_{\kappa} > 4\kappa\right] + \mathbb{E}\left[U_{i}(s^{*}) \mid Z_{\kappa} \leq 4\kappa\right] \Pr\left[Z_{\kappa} \leq 4\kappa\right] - \mathbb{E}\left[U_{i}(\hat{s})\right]$$

$$= \mathbb{E}\left[U_{i}(s^{*}) \mid Z_{\kappa} > 4\kappa\right] \Pr\left[Z_{\kappa} > 4\kappa\right] + \mathbb{E}\left[U_{i}(\hat{s}) \mid Z_{\kappa} \leq 4\kappa\right] \Pr\left[Z_{\kappa} \leq 4\kappa\right] - \mathbb{E}\left[U_{i}(\hat{s})\right]$$

$$= \left(\mathbb{E}\left[U_{i}(s^{*}) \mid Z_{\kappa} > 4\kappa\right] - \mathbb{E}\left[U_{i}(\hat{s}) \mid Z_{\kappa} > 4\kappa\right]\right) \Pr\left[Z_{\kappa} > 4\kappa\right]$$

$$= \left(2^{\kappa} - 1\right) \Pr\left[Z_{\kappa} > 4\kappa\right]$$

$$= \left(2^{\kappa} - 1\right) \left(1 - \alpha^{3}\right)^{\kappa} \qquad \left(\Pr\left[Z_{\kappa} \leq 4\kappa\right] = 1 - (1 - \alpha^{3})^{\kappa}\right)\right)$$

$$= \left(2^{\kappa} - 1\right) \left(\frac{26}{27}\right)^{\kappa}. \qquad \left(\alpha = \frac{1}{3}\right)$$

Setting $\kappa_1 = \max\{2, \lceil \frac{\ln \epsilon}{\ln 52/27} \rceil\}$, where $\lceil \cdot \rceil$ is the upper rounding operation, one can check for every $\kappa > \kappa_1$,

$$\mathbb{E}\left[U_i(s^*)\right] > \mathbb{E}\left[U_i(\hat{s})\right] + \epsilon$$

Therefore, the strategy $\hat{s}(1^{\kappa})$ is not a ϵ -Nash equilibrium of the HT game.

Corollary 6.2. The strategy \hat{s} (shown in Table 3) is not a computational Nash equilibrium of the HT game, shown in Table 1.

Proof. We start by defining a protocol \tilde{s} for the HT game. \tilde{s} follows exactly the stopping strategy given by \hat{s} , but enforces termination after 8κ rounds, instead of the 4κ rounds enforced in \hat{s} . One can check that both protocols \tilde{s} and \hat{s} run in polynomial-time (in terms of the parameter κ).

To prove the statement, it suffices to show that there exists a positive integer $\kappa_1 \in \mathbb{N}$ and a polynomial function $poly(\cdot)$ such that for every $\kappa \geq \kappa_1$, the following holds

$$\mathbb{E}\left[U_i(\tilde{s})\right] - \mathbb{E}\left[U_i(\hat{s})\right] > \frac{1}{\operatorname{poly}\left(\kappa\right)}$$

Let Z_{κ} be the random variable denoting number of rounds until the protocol \tilde{s} terminates. One can check

$$\mathbb{E} [U_i(\tilde{s})] - \mathbb{E} [U_i(\hat{s})]$$

$$= \mathbb{E} [U_i(\tilde{s}) \mid Z_{\kappa} > 4\kappa] \Pr [Z_{\kappa} > 4\kappa] + \mathbb{E} [U_i(\tilde{s}) \mid Z_{\kappa} \le 4\kappa] \Pr [Z_{\kappa} \le 4\kappa] - \mathbb{E} [U_i(\hat{s})]$$

$$= \mathbb{E} [U_i(\tilde{s}) \mid Z_{\kappa} > 4\kappa] \Pr [Z_{\kappa} > 4\kappa] + \mathbb{E} [U_i(\hat{s}) \mid Z_{\kappa} \le 4\kappa] \Pr [Z_{\kappa} \le 4\kappa] - \mathbb{E} [U_i(\hat{s})]$$

$$= (\mathbb{E} [U_i(\tilde{s}) \mid Z_{\kappa} > 4\kappa] - \mathbb{E} [U_i(\hat{s}) \mid Z_{\kappa} > 4\kappa]) \Pr [Z_{\kappa} > 4\kappa]$$

$$= \left(\mathbb{E}\left[U_{i}(\tilde{s}) \mid Z_{\kappa} > 4\kappa\right] - 1\right) \Pr\left[Z_{\kappa} > 4\kappa\right]$$

$$= \mathbb{E}\left[U_{i}(\tilde{s}) \mid Z_{\kappa} > 8\kappa\right] \Pr\left[Z_{\kappa} > 8\kappa\right] + \mathbb{E}\left[U_{i}(\tilde{s}) \mid 4\kappa \leq Z_{\kappa} \leq 8\kappa\right] \Pr\left[4\kappa \leq Z_{\kappa} \leq 8\kappa\right] - \Pr\left[Z_{\kappa} > 4\kappa\right]$$

$$= \left(2^{\kappa} - 1\right) \Pr\left[4\kappa \leq Z_{\kappa} \leq 8\kappa\right]$$

$$= \left(2^{\kappa} - 1\right) \left(\left(1 - \alpha^{3}\right)^{\kappa} - \left(1 - \alpha^{3}\right)^{2\kappa}\right)$$

$$= \left(2^{\kappa} - 1\right) \left(\left(\frac{26}{27}\right)^{\kappa} \left(1 - \left(\frac{26}{27}\right)^{\kappa}\right)\right),$$

which is not a negligible function.

6.2 Robustness of pseudo-equilibrium

In this example, we consider a single-item second-price auction with n bidders. The bidders have values $\mathbf{v} = (v_1, \ldots, v_n)$ such that $v_i \in [0, V]$ for all i. In the auction, each bidder i submits a sealed bid b_i . The highest bidder wins the item and pays the second-highest bid. Let x, p be the allocation and the payment rule of the auction. All agents are quasi-linear, i.e., the utility of agent i is $u_i(\mathbf{b}) = v_i \cdot x_i(\mathbf{b}) - p_i(\mathbf{b})$. Second-price auction is a truthful auction. Let $i^* = \arg \max_i \{v_i\}$ be the winner of the auction.

We implement this auction using cryptographic commitments. We will consider two settings, one that uses ideal commitments C and the other that uses cryptographic commitments. Let \mathcal{G}^{C} (\mathcal{G}^{Π}) be the auction with ideal (cryptographic) commitments. We break the protocol into the following three rounds.

We overload the notation and use Com(b) to denote the action of committing to the value b. In $\mathcal{G}^{\mathbb{C}}$, this is done by sending a message to the ideal functionality C. The ideal functionality records the value b and the identity of the agent who sent it, and then it sends a "receipt" to the auctioneer. When a player sends "open" to C it sends the value b and the identity of the player to the auctioneer. On the other hand, in \mathcal{G}^{Π} , Com(b) is implemented with a cryptographic commitment (e.g., ElGamal-based commitments or any other computationally hiding and perfectly binding commitment scheme). That is c = Com(b) = Commit(b, r) where the function $Commit(\cdot, \cdot)$ is specified by the protocol and r is selected uniformly at random. A player can "open" the commitment by sending b, r to the auctioneer.

Formally, we consider the following game.

 $\underline{\text{Round } 1}$:

- Each bidder picks a bid, b_i , and sends $c_i = Com(b_i)$ to the auctioneer.
- The auctioneer sends $C = \{c_1, \ldots, c_n\}$ back to the players.

Round 2:

• Each bidder opens the commitment to the auctioneer.

<u>Auction</u>:

- The auctioneer selects all valid bids, **b** and runs the auctions $x(\mathbf{b})$ and $p(\mathbf{b})$.
- The auctioneer sends to the winner of the auction the amount of the payment and \perp to all other bidders.

First, we can easily verify that playing truthfully is a Nash equilibrium in the ideal auction \mathcal{G}^{C} . Intuitively, suppose that there is a strategy that achieves higher utility for some agent *i*. Notice that in the ideal game, a player can only interact with the ideal functionality by sending a bid *b* or the message "open". Since the commitments are ideal, there is no way that a commitment can be opened to a different value. Therefore, the bid *b* would also increase the utility in the auction without commitments. That is impossible, given that the second-price auction is truthful.

A natural assumption about players, especially over the Internet, is that they gain utility from any additional information they collect. In this case, the only information that a participant gets is either that they won or lost the auction and no more information. For a player, the identity of the highest bidder is the information the player seeks. We consider the vector $x_i = (x_{i,1}, \ldots, x_{i,n})$ to be a probability distribution that bidder *i* assigns over the identity of the highest bidder. That is, $x_{i,j}$ represents how much bidder *i* believes that bidder *j* won.

We modify the utility function to reflect the preferences of the players. A bidder, in addition to the utility of the auction, gains a reward R with probability $q_i = \max\left\{x_{i,i^*} - \frac{1}{n-1}, 0\right\}$. The probability q_i represents how much better the probability of guessing is compared to random guessing the winner.

First, in the ideal game $\mathcal{G}^{\mathbb{C}}$ we know that ideal commitments do not leak any information. Therefore, for any $i \neq i^*$ no strategy achieves $q_i > 0$. On the other hand, in the real game \mathcal{G}^{Π} we observe that q_i is strictly positive for a bidder who attempts to break the commitment scheme. Thus, a bidder can increase their expected utility by improving their probability of obtaining the additional reward R.

We consider the following profiles

- Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be the strategy profile where all players bid truthfully and each agent *i* assign $x_{i,j} = \frac{1}{n-1}$ for all $j \neq i$ (agent *i*^{*} assigns x_{i^*,i^*}).
- Let s be the corresponding strategy of σ in \mathcal{G}^{Π} (i.e., every agent follows the same steps as in σ except commits and opens as specified in the protocol).
- Let \hat{s} be the strategy in which all agents bid truthfully and additionally try to break the commitments.
- Let \tilde{s} be the strategy in which all agents bid truthfully except $i \neq i^*$. Agent *i* overbids and additionally tries to break the commitments.

As we mentioned earlier, for all *i* when playing according to \hat{s} we have $q_i > 0$. Thus, $\mathbb{E}[U_i(\hat{s})] > \mathbb{E}[U_i(s)] = 0$. That shows that playing truthfully is not Nash.

We can see that for all $i \neq i^*$ the probability $q_i \leq \operatorname{negl}(\kappa)$, where κ is the security parameter of the commitment protocol. Suppose for contradiction that q_i is non-negligible. This means that bidder *i* can correctly guess the winner with a probability significantly larger than 1/(n-1). Since the only information available comes from the commitment scheme itself, this implies that bidder *i* has extracted information about the bids from the commitments. However, the commitments are computationally hiding, meaning no polynomial time algorithm should be able to find the committed value with non-negligible probability.

However, such strategy does not change the outcome of the auction a lot. Consider an agent $i \neq i^*$. Depending on the value of κ and the reward R the extreme deviation strategy \tilde{s} where i intentionally overbids to get the value of the true highest bidder, since now this is the value she needs to pay for the item. In this case, the probability of receiving the reward is q'_i which is strictly better than q_i when playing according to \hat{s} . Even more surprisingly, this strategy depends entirely on the reward R, the value of κ and the probability q'_i . However, all we know for q'_i is that is negligible but it's impossible to find a closed formula. That makes the analysis of the real game infeasible.

Finally, we will show that bidding truthfully is in fact a pseudo-equilibrium. In the ideal game \mathcal{G}^{C} , the utilities do not depend on any parameter; hence we can apply Theorem 3.1 to show that σ is also Pseudo in the ideal game. Finally, we can apply Theorem 5.1 to get that s (the corresponding strategy of σ) is a pseudo-equilibrium of the real game \mathcal{G}^{Π} .

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A HT game

A.1 Definition and Protocols

HT Game

- Notations. There are three players P_1, P_2, P_3 . For a player index $i \in \{1, 2, 3\}$, let i^+ denote i + 1, except that 3^+ is 1. Similarly, let i^- denote i 1, except that 1^- wraps around to 3. For ease of understanding, and whenever it is not ambiguous, we refer to Player P_{i^+} as the *right neighbor* of Player P_i , and Player P_{i^-} as the *left neighbor* of Player P_i .
- Game Setup. Each player P_i holds an infinite sequence of shares $\{s_i^1, s_i^2, \ldots\}$, where $s_i^j \in \{0, 1\}^{\operatorname{poly}(\kappa)}$. For *j*-th element, where $j \in \{1, 2, 3, \ldots\}$, the tuple (s_1^j, s_2^j, s_3^j) forms a 3-out-of-3 secret sharing of the secret m^* . Shares in the sequence $\{s_i^1, s_i^2, \ldots\}$ are mutually independent, i.e., s_i^j is independent of s_i^k for all $j \neq k$.
- Actions per Round. In the round $t \in \{1, 2, ...\}$, each player P_i chooses an action tuple (l_i^t, r_i^t) , where l_i^t and r_i^t represent the actions of sending a message to the *left neighbor* (Player P_{i^-}) and the *right neighbor* (Player P_{i^+}), respectively. Each action $l_i^t, r_i^t \in \{0, 1\}^{\mathsf{poly}(\kappa)} \cup \{\emptyset, \bot\}$ can be one of the following:
 - $-m_i^j \in \{0,1\}^{\operatorname{poly}(\kappa)}$: Send a polynomial-length message.
 - $\varnothing:$ Send a null message.
 - \perp : Abort the game immediately.

All action tuples (l_i^t, r_i^t) are chosen simultaneously by all players at the start of each round.

- Outcome of a Round. At the end of round t, each player P_i observes the tuple (r_{i-}^t, l_{i+}^t) , where r_{i-}^t is the message received from its *left neighbor* (Player P_{i-}), and l_{i+}^t is the message received from its *right neighbor* (Player P_{i+}). Then:
 - If at least one player can reconstruct m^* from the messages it received and its own shares, i.e., m^* can be constructed from $\{(r_{i-}^j, l_{i+}^j), s_i^j\}_{j \le t}$, the game ends.
 - If at least one player chooses to abort (\perp) , the game also ends.
 - Otherwise, the game proceeds to the next round.
- Utilities of the Game. All players share the same utility function. Let T denote the final round index. For each player P_i , define $\inf_i = \{(r_{i-}^j, l_{i+}^j), s_i^j\}_{j \leq T}$ as the set of all information available to Player P_i , which includes all messages received from other players during the T rounds and its own shares. Let $\operatorname{rec}_{m^*} : \{0, 1\}^* \to \{0, 1\}$ be the reconstruction function for the secret m^* . The function rec_{m^*} takes as input arbitrary message and outputs 1 if it can successfully reconstruct m^* ; otherwise, it outputs 0. Then, the utility of each player P_i , denoted as u_i , is determined as follows:
 - If only player *i* learns the secret m^* , i.e., $\operatorname{rec}_{m^*}(\operatorname{info}_i) = 1$ and $\operatorname{rec}_{m^*}(\operatorname{info}_{i^+}) = \operatorname{rec}_{m^*}(\operatorname{info}_{i^-}) = 0$, then $u_i = 2^{\kappa+1} + 2$.
 - If all players learn the secret m^* , i.e., $\operatorname{rec}_{m^*}(\operatorname{info}_i) = \operatorname{rec}_{m^*}(\operatorname{info}_{i^+}) = \operatorname{rec}_{m^*}(\operatorname{info}_{i^-}) = 1$, then $u_i = 2^{\kappa}$.
 - If no player learns the secret m^* , i.e., $\operatorname{rec}_{m^*}(\operatorname{info}_i) = \operatorname{rec}_{m^*}(\operatorname{info}_{i^+}) = \operatorname{rec}_{m^*}(\operatorname{info}_{i^-}) = 0$, then $u_i = 1$.

Table 1: Secret Sharing Game (HT Game) [HT04]

Strategy s^* for HT Game

- Actions in Rounds of the Form t = 4j + 1. For each round t = 4j + 1, where $j \in \{0, 1, 2, ...\}$, every player P_i does the following:
 - Flip a biased coin dc_i^j with parameter $\alpha = \frac{1}{3}$, i.e., $dc_i^j \sim \text{Bernoulli}(\alpha)$. We call dc_i^j the *j*-th decision coin of player P_i .
 - Flip a fair coin mc_i^j , i.e., $mc_i^j \sim \text{Bernoulli}(\frac{1}{2})$. We call mc_i^j the *j*-th mask of player P_i .
 - Send the masked decision coin to the *left neighbor*, i.e., Set $l_i^t \leftarrow dc_i^j \oplus mc_i^j$.
 - Send the mask to the *right neighbor*, i.e., Set $r_i^t \leftarrow mc_i^j$.
- Actions in Rounds of the Form t = 4j + 2. For each round t = 4j + 2, where $j \in \{0, 1, 2, ...\}$, every player P_i does the following:
 - Send the masked double-decision coin $dc_i^j \oplus l_{i+}^{t-1} = dc_i^j \oplus dc_{i+}^j \oplus mc_{i+}^j$ to the left neighbor, where dc_i^j is player P_i 's *j*-th decision coin, and $l_{i+}^{t-1} = dc_{i+}^j \oplus mc_{i+}^j$ is the *j*-th masked decision coin received from its right neighbor P_{i+} . That is, Set $l_i^t \leftarrow dc_i^j \oplus l_{i+}^{t-1}$.
 - Send a null message to the right neighbor, i.e., Set $r_i^t \leftarrow \emptyset$.
- Actions in Rounds of the Form t = 4j + 3. For each round t = 4j + 3, where $j \in \{0, 1, 2, ...\}$, every player P_i does the following:
 - Compute the *j*-th group decision bit $p^j = dc_1^j \oplus dc_2^j \oplus dc_3^j$, where dc_1^j, dc_2^j, dc_3^j are the *j*-th decision coin of player P_1, P_2, P_3 . *p* can be computed from its passed three round message $l_{i^+}^{t-1} \oplus r_{i^-}^{t-2} \oplus dc_i^j = dc_{i^+}^j \oplus dc_{i^-}^j \oplus mc_{i^-}^j \oplus mc_{i^-}^j \oplus dc_i^j = dc_1^j \oplus dc_2^j \oplus dc_3^j = p^j$
 - If both the group decision and its own decision for this round is to share, i.e., $p^j = dc_i^j = 1$, then set $r_i^t \leftarrow s_i^j$ and $l_i^t \leftarrow s_i^j$; otherwise set $r_i^t \leftarrow \emptyset$ and $l_i^t \leftarrow \emptyset$.
- Actions in Rounds of the Form t = 4j + 4. For each round t = 4j + 4, where $j \in \{0, 1, 2, ...\}$, every player P_i does the following:
 - If the *j*-th group decision is not to share and player P_i receives no shares from others, i.e., $p^j = 0$ and $l_{i^+}^{t-1} = r_{i^-}^{t-1} = \emptyset$; OR
 - both the *j*-th group decision and P_i 's decision are to share and player P_i receives no shares from others, i.e., $p^j = dc_i^j = 1$ and $l_{i^+}^{t-1} = r_{i^-}^{t-1} = \emptyset$; OR
 - the *j*-th group decision is to share, player P_i 's decision is not to share and player P_i receives exactly one share from others, i.e., $p^j = 1, dc_i^j = 0$ and either $(l_{i^+}^{t-1} = \varnothing, r_{i^-}^{t-1} = s_{i^-}^j)$ or $(l_{i^+}^{t-1} = s_{i^+}^j, r_{i^-}^{t-1} = \varnothing)$, then player P_i requests the game to continue by setting $r_i^t \leftarrow \varnothing$ and $l_i^t \leftarrow \varnothing$.
 - Otherwise P_i aborts the game by $r_i^t \leftarrow \bot$ and $l_i^t \leftarrow \bot$.

Table 2: Strategy s^* for HT Game

Strategy \hat{s} for HT Game

- Actions in Rounds of the Form t = 4j + 1. For each round t = 4j + 1, where $j \in \{0, 1, 2, ..., \kappa\}$, every player P_i does the following:
 - Flip a biased coin dc_i^j with parameter $\alpha = \frac{1}{3}$, i.e., $dc_i^j \sim \text{Bernoulli}(\alpha)$. We call dc_i^j the *j*-th decision coin of player P_i .
 - Flip a fair coin mc_i^j , i.e., $mc_i^j \sim \text{Bernoulli}(\frac{1}{2})$. We call mc_i^j the *j*-th mask of player P_i .
 - Send the masked decision coin to the *left neighbor*, i.e., Set $l_i^t \leftarrow dc_i^j \oplus mc_i^j$.
 - Send the mask to the *right neighbor*, i.e., Set $r_i^t \leftarrow mc_i^j$.
- Actions in Rounds of the Form t = 4j + 2. For each round t = 4j + 2, where $j \in \{0, 1, 2, ..., \kappa\}$, every player P_i does the following:
 - Send the masked double-decision coin $dc_i^j \oplus l_{i^+}^{t-1} = dc_i^j \oplus dc_{i^+}^j \oplus mc_{i^+}^j$ to the *left neighbor*, where dc_i^j is player P_i 's *j*-th decision coin, and $l_{i^+}^{t-1} = dc_{i^+}^j \oplus mc_{i^+}^j$ is the *j*-th masked decision coin received from its right neighbor P_{i^+} . That is, Set $l_i^t \leftarrow dc_i^j \oplus l_{i^+}^{t-1}$.
 - Send a null message to the right neighbor, i.e., Set $r_i^t \leftarrow \emptyset$.
- Actions in Rounds of the Form t = 4j + 3. For each round t = 4j + 3, where $j \in \{0, 1, 2, ..., \kappa\}$, every player P_i does the following:
 - Compute the *j*-th group decision bit $p^j = dc_1^j \oplus dc_2^j \oplus dc_3^j$, where dc_1^j, dc_2^j, dc_3^j are the *j*-th decision coin of player P_1, P_2, P_3 . *p* can be computed from its passed three round message $l_{i^+}^{t-1} \oplus r_{i^-}^{t-2} \oplus dc_i^j = dc_{i^+}^j \oplus dc_{i^-}^j \oplus mc_{i^-}^j \oplus mc_{i^-}^j \oplus dc_i^j = dc_1^j \oplus dc_2^j \oplus dc_3^j = p^j$
 - If both the group decision and its own decision for this round is to share, i.e., $p^j = dc_i^j = 1$, then set $r_i^t \leftarrow s_i^j$ and $l_i^t \leftarrow s_i^j$; otherwise set $r_i^t \leftarrow \emptyset$ and $l_i^t \leftarrow \emptyset$.
- Actions in Rounds of the Form t = 4j + 4. For each round t = 4j + 4, where $j \in \{0, 1, 2, ..., \kappa\}$, every player P_i does the following:
 - If the *j*-th group decision is not to share and player P_i receives no shares from others, i.e., $p^j = 0$ and $l_{i^+}^{t-1} = r_{i^-}^{t-1} = \emptyset$; OR
 - both the *j*-th group decision and P_i 's decision are to share and player P_i receives no shares from others, i.e., $p^j = dc_i^j = 1$ and $l_{i^+}^{t-1} = r_{i^-}^{t-1} = \emptyset$; OR
 - the *j*-th group decision is to share, player P_i 's decision is not to share and player P_i receives exactly one share from others, i.e., $p^j = 1$, $dc_i^j = 0$ and either $(l_{i^+}^{t-1} = \emptyset, r_{i^-}^{t-1} = s_{i^-}^j)$ or $(l_{i^+}^{t-1} = s_{i^+}^j, r_{i^-}^{t-1} = \emptyset)$, then player P_i requests the game to continue by setting $r_i^t \leftarrow \emptyset$ and $l_i^t \leftarrow \emptyset$.
 - Otherwise P_i aborts the game by setting $r_i^t \leftarrow \bot$ and $l_i^t \leftarrow \bot$.
- Actions in Round $t = 4\kappa + 5$. Every player P_i aborts the game by setting $r_i^t \leftarrow \bot$ and $l_i^t \leftarrow \bot$.

Table 3: Strategy \hat{s} for HT Game

B Computational Nash Equilibrium

For readers interested in a closer comparison between our proposed *pseudo-equilibrium* notion and *computational* Nash equilibrium [HPS16] — a recent and closely related attempt to bridge the gap between cryptography and game theory, in this section, we revisit the definition of computational Nash equilibrium and the necessary context for using it.

Definition B.1 states the formal definition of a computable uniform sequence of games (also referred to as a computational game), which is the specific class of games on which computational Nash equilibrium is defined.

Definition B.1 (Computable Uniform Sequence of Games, Definition 3.1 in [HPS16]).

Let $[c] = \{1, 2, ..., c\}$ denote the set of player indices. A computable uniform sequence (or computational game) $\mathcal{G} = \{G_1, G_2, ...\}$ of games is a sequence of extensive-form game G_i that satisfies the following conditions:

- All the games G_1, G_2, \ldots in the sequence \mathcal{G} involve the same set of players, i.e., $\{1, \cdots, c\}$.
- The histories set H_n for the n-th game G_n has the following properties:
 - Every action available at a non-terminal history is polynomial-size describable. Formally, there exists a polynomial p such that, for all non-terminal histories $h \in H_n^{NT}$, the set of all action A(h) at the history h is at most size p(n): $A(h) \subseteq \{0,1\}^{\leq p(n)}$.
 - It is efficient to check if a sequence of actions (i.e., a history) is valid in G_n . Formally, there exists a PPT algorithm $\mathcal{A} : \{0,1\}^* \mapsto \{0,1\}$ such that, for any history h, $\mathcal{A}(1^n,h) = 1$ if $h \in H_n$, and $\mathcal{A}(1^n,h) = 0$ otherwise.
- It is efficient to compute which player moves at a given valid non-terminal history. Formally, there exists a PPT algorithm $P : \{0,1\}^* \mapsto [c]$ such that, for any history $h \in H_n^{NT}$, it can compute $P(1^n, h) \in [c]$ correctly.
- It is efficient to compute the utility for each player. Formally, for every $i \in [c]$, there exists a PPT utility function $u_i : \{0,1\}^* \mapsto \mathsf{R}$ such that, for every terminal history $h \in H_n^T$, it can compute $u_i(1^n, h)$.

Definition B.2 presents the formal definition of computational Nash equilibrium.

Definition B.2 (Computational Nash Equilibrium, Definition 4.1 in [HPS16]). Let $\mathcal{G} = \{G_1, G_2, ...\}$ be a computable uniform sequence of games. Then, the polynomial-time strategy profile $\vec{M} = \{M_1, ..., M_c\}$ is a computational Nash equilibrium of \mathcal{G} if, for all players $i \in [c]$ and all polynomial-time strategies M'_i in \mathcal{G} for player i, there exists a negligible function ϵ , such that for all n,

$$\sum_{h \in H_n^T} \psi_{\vec{M}}^{G_n}(h) u_i(h) \ge \sum_{h \in H_n^T} \psi_{(M_i', \vec{M}_{-i})}^{G_n}(h) u_i(h) - \epsilon(n),$$

where H_n^T denotes the set of terminal histories in the n-th game G_n , $\psi_{\vec{M}}^{G_n}(\cdot)$ and $\psi_{(M'_i,\vec{M}-i)}^{G_n}(\cdot)$ are the probability distributions over H_n^T induced by playing the strategy profiles \vec{M} and $(M'_i, \vec{M}-i)$, respectively, and $u_i(\cdot)$ is the utility function of player *i*.

To analyze the computational Nash equilibrium, rather than reasoning directly from the definition — which is often challenging due to the complexity of the equilibrium's definition and computational games — the authors propose studying the strategies of the underlying game (or ideal game) that the computational game (or real game) represents. Definition B.3 states necessary conditions that such an underlying game must satisfy, and formally define what it means for a computational game to represent a standard extensive-form game.

Definition B.3 (Represented Underlying Game, Definition 3.3 in [HPS16]). Let $\mathcal{G} = \{G_1, G_2, \ldots\}$ be a computable uniform sequence and G be a fixed extensive-form game. Denote by $f = \{f_1, f_2, \ldots\}$ a sequence of mappings such that each $f_n : H_n \mapsto H$ maps histories (H_n) in the n-th game G_n to histories (H) in the ideal game G. Denote by $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots\}$ a sequence of mappings such that each \mathcal{F}_n maps every strategy σ in the ideal game G to a corresponding strategy in the n-th game G_n . We say that a computational game \mathcal{G} (or real game) $\langle f, F \rangle$ -represents the underlying game (or ideal game) G — or equivalently, that the real game \mathcal{G} is $\langle f, F \rangle$ -corresponding to the ideal game G — if the quadruple $(\mathcal{G}, G, f, \mathcal{F})$ satisfies the following properties:

- UG1. G and all the games G_1, G_2, \ldots in the sequence \mathcal{G} involve the same set of players, i.e., $\{1, \cdots, c\}$.
- UG2. The computational game \mathcal{G} should have the same structure of the ideal game G. More concretely, there exists a sequence of mappings $f = \{f_1, f_2, \ldots\}$ such that each $f_n : H_n \mapsto H$ maps histories (H_n) in the n-th game G_n to histories (H) in the ideal game G, and each f_n is surjective. Furthermore, the surjective mapping f_n has the following properties:
 - (a) Any history $h \in H_n$ in the computational game G_n and its corresponding history $f_n(h) \in H$ in the ideal game G should contain the same number of actions. Formally, $|h| = |f_n(h)|$.
 - (b) For any history $h \in H_n$ and its corresponding history $f_n(h) \in H$, the same player is assigned to move at both h and $f_n(h)$; and

- (c) if h' is a subhistory of h, then $f_n(h')$ is a subhistory of $f_n(h)$; and
- (d) if h and h' are in the same information set in G_n , then $f_n(h)$ and $f_n(h')$ are in the same information set in G.
- (e) For any distinct histories $h, h' \in H_n$, if h and h' are in the same information set in the game G_n , then for any action a such that $h||a \in H_n$ (i.e., action a is valid after history h), the last action in the corresponding mapped histories must be the same; i.e., $LA(f_n(h||a)) = LA(f_n(h'||a))$.
- UG3. For any terminal history $h \in H_n^T$ and its corresponding terminal history $f_n(h) \in H^T$, the utility of each player $i \in [c]$ must be the same in both h and $f_n(h)$, i.e., $u_i(h) = u_i(f_n(h))$.
- UG4. Every strategy in the ideal game G, along with any unilateral deviation from it, can be faithfully simulated by a corresponding polynomial-time strategy (and deviation) in the computational game \mathcal{G} . More concretely, there exists a sequence of mappings $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots\}$ such that each \mathcal{F}_n maps every strategy σ in the ideal game G to a corresponding strategy in the n-th game G_n . Furthermore, the total mapping \mathcal{F}_n has the following properties:
 - (a) The mapping \mathcal{F}_n applies independently to each player's strategy in the game G. Formally, for every strategy profile $\vec{\sigma} = \{\sigma_1, \dots, \sigma_c\}$ in G, its corresponding strategy profile $\mathcal{F}_n(\vec{\sigma})$ in the game G_n has the form $\mathcal{F}_n(\vec{\sigma}) = (\mathcal{F}_n(\sigma_1), \dots, \mathcal{F}_n(\sigma_c))$.
 - (b) For every strategy σ_i of player $i \in [c]$ in the ideal game G, there exists a PPT strategy M^{σ_i} in the computational game G that simulates the behavior of σ_i . More concretely, let r_n and v_n denote the randomness and view of player i when playing the corresponding strategy $\mathcal{F}_n(\sigma_i)$ in the computational game G_n . The algorithm M^{σ_i} takes $(1^n, v_n, r_n)$ as input and outputs the next action for player i. For every strategy σ_i , there must exist a PPT algorithm M^{σ_i} such that its output equals the last action in the history $f_n(\mathcal{F}_n(\sigma_i)(v_n, r_n)) \in H$ that is, the corresponding ideal-game history obtained by mapping the computational history (produced by playing $\mathcal{F}_n(\sigma_i)$ with view v_n and randomness r_n) back to the ideal game via f_n . Formally,

$$LA(f_n(\mathcal{F}_n(\sigma_i)(v_n, r_n))) = M^{\sigma_i}(1^n, v_n, r_n).$$

(c) For all strategy profiles $\vec{\sigma}$ in G, all players *i*, and all polynomial-time strategies M'_i for player *i* in \mathcal{G} , there exists a sequence $\{\sigma'_1, \sigma'_2, \ldots\}$ of strategies for player *i* in G such that

$$\left\{\phi_{M'_i,\mathcal{F}(\vec{\sigma}_{-i})}^{G_n}\right\}_n \text{ is computationally indistinguishable from } \left\{\rho_{(\sigma'_n,\vec{\sigma}_{-i})}^G\right\}_n,$$

where H_n^T and H^T denote the sets of terminal histories in the n-th computational game G_n and the ideal game G, respectively; and the distribution $\phi_{M'_i,\mathcal{F}(\vec{\sigma}-i)}^{G_n}$ is the probability distribution over H_n^T induced by playing the strategy profiles $(M'_i,\mathcal{F}(\vec{\sigma}-i))$ in G_n ; and similarly, $\rho(\sigma'_n,\vec{\sigma}-i)^G$ is the probability distribution over H^T induced by playing the strategy profiles $(\sigma'_n,\vec{\sigma}-i)$ in G.

Finally, the following theorem shows how a computational Nash equilibrium in the computational game \mathcal{G} can be obtained from an underlying game G and a corresponding strategy mapping \mathcal{F} .

Theorem B.1 (Theorem 4.2 in [HPS16]). If the computational game $\mathcal{G} \langle f, \mathcal{F} \rangle$ -represents the ideal game G and $\vec{\sigma}$ is an Nash equilibrium in G, then $\mathcal{F}(\vec{\sigma})$ is a computational Nash equilibrium of \mathcal{G} .

B.1 Guessing Game

We are revisiting a slightly modified version of the two-player game presented in Example 1.2. The game is zerosum, player 1 commits to κ bits, and player 2 needs to guess the string. If player 2 correctly guesses all κ bits, then she gets utility of 2^{κ} ; otherwise, player 1 wins and player 2 gets utility of 0. Formally, we consider the games shown in Table 4.

Ideal Game	Real Game
 Commit Phase: Player 1 selects x ∈ {0,1}^κ and commits it using an ideal commitment. Guess Phase: Player 2 chooses a guess y ∈ {0,1}^κ. Reveal Phase: Player 1 can open the commitment to reveal x. If player 1 does not open the commitment, she receives utility −1. 	 Commit Phase: Player 1 chooses x ∈ {0,1}^κ and a random string r, then sends to Player 2 c = Commit(1^κ, x; r). Guess Phase: Player 2 outputs a guess y based on the received c. Reveal Phase: Player 1 sends (x, r) to open the commitment. If Open(1^κ, x, r) verifies, the payoffs are assigned as in the ideal game.
For both games, the utilities of the player are $u_1 = -u_2 = 0$ if $x \neq y$ and $u_1 = -u_2 = -2^{\kappa}$ if $x = y$.	

Table 4: Guessing Game

In the idea game, it is easy to show that the strategy profile where x, y are chosen uniformly at random is a Nash equilibrium. To see this, we will show that there is no profitable deviation for player 2. By symmetry, the same holds for Player 1. Notice that the utility random variable of player 2 is $U_2 = \{0 \text{ w.p. } 1 - 1/2^{\kappa} \& 2^{\kappa} \text{ w.p. } 1/2^{\kappa}\}$ with expected utility $\mathbb{E}[U_2] = 1$. It suffices to show that for every unilateral deviation — whose utility random variable is denoted by U'_2 — it holds that $\mathbb{E}[U_2] \ge \mathbb{E}[U'_2]$. Let X be the uniform distribution over $\{0,1\}^{\kappa}$, and player 1 selects x according to X. Under an arbitrary unilateral deviation by player 2, let Y denote the (potentially non-uniform) distribution from which player 2 selects y. Then player 2's expected utility would be

$$\mathbb{E}\left[U_{2}'\right] = \mathbb{E}\left[u_{2}(X,Y)\right] = \sum_{x \in \{0,1\}^{\kappa}} 2^{\kappa} \cdot \Pr\left[X=x\right] \Pr\left[Y=x\right] = \sum_{x \in \{0,1\}^{\kappa}} 2^{\kappa} \cdot \frac{1}{2^{\kappa}} \Pr\left[Y=x\right] = \sum_{x \in \{0,1\}^{\kappa}} \Pr\left[Y=x\right] = 1.$$

In the real game, one can argue that the uniform strategy is a pseudo-equilibrium. This follows directly from our main theorem (Theorem 5.1). To apply Theorem 5.1, we first observe that the uniform strategy is a Nash equilibrium in the ideal game implies that it is also a pseudo-equilibrium in the ideal game. Furthermore, by Theorem 5.1, since the uniform strategy is a pseudo-equilibrium in the ideal game, and the only difference from the ideal game is that the ideal commitments — which leak no information about x — are replaced with cryptographic commitments — that leak only a negligible amount of information about x to any PPT adversary — it implies that the uniform strategy is a pseudo-equilibrium in the real game.

However, in the real game, it is interesting to see that the uniform strategy (Nash of the ideal game) is not always a computational Nash (Definition B.2). We first notice that the real game is a computable sequence of games, on which the computational Nash is defined. This is by definition (Definition B.1): the real game always involves two players, has two rounds, and the utility can be efficiently computed. For a strategy to be a computational Nash, for all players, there is no unilateral deviation that can increase the expected utility by a noticeable amount. To prove the uniform strategy a computational Nash, for every PPT unilateral deviation and its corresponding utility random variable, denoted by U_2^* , there must exists a negligible function ϵ such that

$$\mathbb{E}\left[U_2\right] \ge \mathbb{E}\left[U_2^*\right] - \epsilon(\kappa),$$

where U_2 is the utility random variable for player 2 when playing the uniform strategy. Now consider a unilateral deviation by Player 2, a polynomial-time strategy that attempts to break the cryptographic commitment scheme to extract x. By the definition of the cryptographic commitment scheme, any such a PPT strategy can only increase the success probability (i.e., the probability that y == x) by at most a negligible function $\mu(\kappa)$. Therefore, its corresponding utility random variable has the form of $U_2^* = \{0 \text{ w.p. } 1 - 1/2^{\kappa} - \mu(\kappa) \& 2^{\kappa} \text{ w.p. } 1/2^{\kappa} + \mu(\kappa)\}$, which leads to

$$\mathbb{E}\left[U_2\right] \ge \mathbb{E}\left[U_2^*\right] - \epsilon(\kappa) \implies 1 \ge 2^{\kappa}(1/2^{\kappa} + \mu(\kappa)) - \epsilon(\kappa).$$

After arrangement, $\mathbb{E}[U_2] \geq \mathbb{E}[U_2^*] - \epsilon(\kappa)$ holds requires that $2^{\kappa} \cdot \mu(\kappa)$ is still negligible. However, this does not necessarily to be true given only the assumption that the real-world commitment scheme is cryptographically

secure. Indeed, many cryptographic commitment schemes have known non-trivial attacks that make $2^{\kappa} \cdot \mu(\kappa)$ non-negligible. For example, hash-based commitment schemes are subject to birthday-bound attacks. Thus, we conclude that the uniform strategy is not necessarily a computational Nash equilibrium. Furthermore, we note that it is not easy to find a strategy profile that is a computational Nash, since this requires analyzing the exact probability of success, and cryptography does not provide such tools.