# Agency Problems and Adversarial Bilevel Optimization under Uncertainty and Cyber Threats

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**Abstract**

We study an agency problem between a holding company and its subsidiary, exposed to cyber threats that affect the overall value of the subsidiary. The holding company seeks to design an optimal incentive scheme to mitigate these losses. In response, the subsidiary selects an optimal cybersecurity investment strategy, modeled through a stochastic epidemiological SIR (Susceptible-Infected-Recovered) framework. The cyber threat landscape is captured through an L-hop risk framework with two primary sources of risk: (i) internal risk propagation via the contagion parameters in the SIR model, and (ii) external cyberattacks from a malicious external hacker. The uncertainty and adversarial nature of the hacking lead to consider a robust stochastic control approach that allows for increased volatility and ambiguity induced by cyber incidents. The agency problem is formulated as a max-min bilevel stochastic control problem with accidents. First, we derive the incentive compatibility condition by reducing the subsidiary's optimal response to the solution of a second-order backward stochastic differential equation with jumps. Next, we demonstrate that the principal's problem can be equivalently reformulated as an integro-partial Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation. By extending the stochastic Perron's method to our setting, we show that the value function of the problem is the unique viscosity solution to the resulting integro partial HJBI equation.

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# 1 Introduction

According to Governor Michael S. Barr, speaking at the Federal Reserve Bank of New York on April 17, 2025 *"Cybercrime is on the rise, and cybercriminals are increasingly turning to Gen AI to facilitate their crimes. Criminal tactics are becoming more sophisticated and available to a broader range of criminals. Estimates of direct and indirect costs of cyber incidents range from 1 to 10 percent of global GDP. Deepfake attacks have seen a twenty-fold increase in the last three years".* Governor Barr's remarks underscore the growing severity of cyber threats fueled by the hyper-connectivity of modern society. Individuals, businesses, public institutions, and critical infrastructure are increasingly interconnected through digital networks—creating vulnerabilities across virtually every sector. From social media platforms and private messaging services to healthcare systems, governments, and financial institutions, no domain is immune. These threats are not geographically confined either; cyberattacks are now a global concern, affecting nations and industries worldwide. Recent geopolitical developments—such as the Russia-Ukraine war—have further intensified cyber threats, particularly across Europe and NATO member states. Likewise, the COVID-19 pandemic, which accelerated the digitalization of services and online interaction, has expanded the attack surface for cybercriminals. However, cyber threats have been growing increasingly sophisticated over the past few decades, making it urgent to develop a strong agenda to address it as one of the main challenge of the 21st century (see, e.g., Tatar et al. (2014); Karabacak and Tatar (2014); Eling et al. (2021); Amin (2019); Ghadge et al. (2020)). To address these challenges, the U.S. Department of Homeland Security's Science and Technology Directorate has launched the Cyber Risk Economics (CyRiE) project. This initiative promotes research into the legal, behavioral, technical, and economic dimensions of cybersecurity. A key component of CyRiE focuses on designing effective incentives to optimize cyber-risk management, aiming to guide organizations in allocating resources toward the most impactful and valuable defenses.

This work contributes to that objective by exploring how a parent (holding) firm can design optimal incentives and compensation mechanisms for its subsidiaries operating under cyber threat conditions. The goal is to ensure efficient monitoring and management of both the subsidiary's portfolio and its cybersecurity strategies. Our proposed model incorporates an essential characteristics of cyber risk in the modern economy: L-hop attack propagation.

## 1.1 Cyber risk modeling and L-hop risk propagation

Cyberattacks vary widely in form and mechanism (see, e.g., Uma and Padmavathi (2013); Hathaway et al. (2012); Hillairet et al. (2023); Grove et al. (2019); Boumezoued et al. (2023); Hillairet et al. (2024)), but L-hop propagation models are particularly useful for capturing the dynamics of both external and internal threats. The term **L-hop** refers to the number of network connections (or "hops") an attack can traverse before reaching its target. External threats originate outside the network—such as direct hacking attempts—modeled using a point process with exogenous intensity. Internal threats emerge from within the network, typically through infected nodes spreading malware or viruses. These internal dynamics are modeled using compartmental epidemiological models, such as the SIR (Susceptible-Infectious-Recovered) framework, see e.g. Capasso (1993); Britton (2010); Elie et al. (2020), in the context of cyber risk (see, e.g., Del Rey et al. (2022); Hillairet et al. (2022, 2024)). By integrating these components, the proposed model offers a robust framework for evaluating how financial firms can design efficient intra-organizational incentives that align cybersecurity investments with the broader objectives of risk mitigation and financial resilience.

## 1.2 Incentives and agency theory

Turning now to incentive mechanism, it has been investigated since the 1960s in economy and known as contract theory or agency problem, model with a Principal-Agent framework with information asymmetry. Holmstrom and Milgrom's 1987 pioneer work Holmstrom and Milgrom (1987) has set the paradigm in a continuous-time framework with continuous controlled process. It has then regained interest in the mathematical community in the last decades with the work of Sannikov Sannikov (2008) and Cvitanic, Possamai and Touzi Cvitanić et al. (2018, 2017). In our model, the holding form (the principal) monitors indirectly the action of the subsidiary (the agent) by proposing a compensation for its activities. The holding firm does not have a direct access to the activities of its subsidiary and only observes the result of its work through its wealth and corrupted devices in the SIR system. This asymmetry of information arises in a moral hazard situation in which the principal must anticipates the best reaction of the agent to propose an optimal incentives scheme. This problem is equivalent to solve a Stackelberg game in continuous time, see for example Li and Sethi (2017); Hernández-Santibáñez (2024); Hernández et al. (2024). We usually address this problem as a bilevel stochastic optimization, in which the problem of the agent is embedded into the problem of the principal, known as *the incentive compatibility condition* of the compensation offers by the principal to the agent ensuring the existence of a best reaction activity, see e.g. Mastrolia and Zhang (2025); Dempe and Zemkoho (2020). We refer to Tirole (2010); Cvitanic and Zhang (2012) for a more detailed overview of principal-agent, Stackelberg games and agency problem.

### 1.3 Model uncertainty and ambiguity

In the realm of cybersecurity, the inherent unpredictability and knowledge gaps that arise when constructing and deploying models to predict or prevent cyber-threats lead to various types of uncertainty. These uncertainties can arise from multiple sources and understanding them is vital for the development of more resilient and adaptive cybersecurity systems. This work focuses on three key types of uncertainty: (1) the propagation of cyber risk within the subsidiary cluster; (2) the impact on the system's wealth; and (3) the randomness and ambiguity inherent in the behavior of cyber attacks. This section introduces informally the problem investigated. A more rigorous framework is provided hereafter.

As discussed previously, the propagation of a cyber attack is modeled using an epidemiological framework with stochastic noise. Specifically, we assume that the spread of the attack within the cluster—referred to as the internal L-hop risk—is governed by the following SIR

(Susceptible-Infected-Recovered) system:

$$\begin{cases} dS_t = (-\beta S_t I_t - \alpha_t S_t - \eta_t S_t)dt - \tilde{\sigma}(t, \alpha_t) I_t S_t d\widetilde{W}_t \\ dI_t = (\beta S_t I_t - \rho I_t + \eta_t S_t)dt + \tilde{\sigma}(t, \alpha_t) I_t S_t d\widetilde{W}_t \\ dR_t = \rho I_t dt + \alpha_t S_t dt, \end{cases}$$

where $\eta$ denotes the unknown cyber attack and $\alpha$ the protection strategy used by the subsidiary. Note that the uncertainty arise by considering that the propagation parameter $\beta$ is random and evolves as follow between time $t$ and $t + dt$

$$d\beta_t \longrightarrow \beta dt + \tilde{\sigma}(t, \eta_t) d\widetilde{W}_t,$$

where $\widetilde{W}$ is a standard Brownian motion and $\tilde{\sigma}$ the volatility induced by the cyber attack $\eta$ propagating in the SIR system.

Regarding the uncertainty in the wealth of the subsidiary, we assume that the portfolio of the firm is given at time $t$ by the solution to the following SDE

$$dP_s = P_s \left( \mu(s, I_s) dt + \sigma(s, I_s, \eta_s) dW_s + \int_E l_s(e) \mu_P(de, ds) \right),$$

where $\mu$ represents the drift of the subsidiary's wealth, $\sigma$ represents the uncertainty induced by the hacking on the financial market impacting the portfolio value of the subsidiary with possible accident given by a Poisson random measures $\mu_P$, which intensity $\lambda$ depends on the compromised devices and the direct hacking activity, reflecting the L-hop modeling. Finally, Cyberattackers continuously evolve their tactics, techniques, and procedures. Attackers may exploit vulnerabilities or create novel attack patterns that were not present in the training data, leading to model uncertainty and ambiguity on their actions $\eta$. This issue is usually addressed by adopting a robust approach of the problem; see, for example, Balter et al. (2023); Bielecki et al. (2014); Hernández-Santibánez and Mastrolia (2019); Mastrolia and Possamaï (2018); Sung (2022). Let $(\eta, \mathbb{P})$ represent a probability model defined by the cyber attack, leading to the formulation of a Stackelberg bilevel stochastic optimization problem, which can be broadly outlined as follows:

$$\begin{cases} V_0^P = \sup_{\xi,\hat{\alpha}} \inf_{(\mathbb{P},\eta)} \mathbb{E}^{\mathbb{P}}[U_P(\xi, P_T, S_T, I_T, C_T, \hat{\alpha}, \eta)], \\ \text{ subject to} \\ (IC - \sigma) \quad V_0^A(\xi) := \sup_{\alpha} \inf_{(\mathbb{P},\eta)} \mathbb{E}^{\mathbb{P}}[U_A(\xi, P_T, S_T, I_T, C_T^A, \alpha)] = \mathbb{E}^{\mathbb{P}^{\hat{\eta}}}[U_A(\xi, P_T, S_T, I_T, C_T^A, \hat{\alpha})] \\ (R) \quad V_0^A(\xi) \geq R_0. \end{cases}$$

We call this problem ($\mathbf{2Mm} - \sigma$) standing for bilevel Max-min optimization with ambiguity, $(IC - \sigma)$ is the incentive compatibility condition with ambiguity, $(R)$ is the reservation utility constraint, $U_P, U_A$ are the utility functions of the holding company and the subsidiary, respectively, $\xi$ represents the compensation proposed to the subsidiary, and $C_T, C_T^A$ represent the additional discontinuous costs incurred by the holding company and the subsidiary, respectively, as a result of cyber attacks.

## 1.4 Comparison with the litterature

We now detail the main contributions of this work on three different topics: cyber risk modeling, stochastic optimization and agency problem and cyber risk economics.

- *Cyber risk modeling and economics.* While most models studied to date have focused on either discrete-time optimization or deterministic SIR models for cyber risk, our approach addresses cyber risk uncertainty through a fully stochastic framework that includes volatility uncertainty in both the SIR system and the wealth process. This extends, for example, the work of Khouzani et al. (2019); Hillairet et al. (2022). In addition, we provide a comprehensive model of L-hop risk propagation using a stochastic SIR system with model ambiguity.
  Incentive mechanisms for cyber risk management have been previously studied in contexts such as health data protection and optimal cybersecurity investments; see Khouzani et al. (2019); Zhang and Malacaria (2021); Wessels et al. (2021); Bauer and Van Eeten (2009); Lee and Aswani (2022). We contribute to this literature by extending the analysis to a continuous-time setting, focusing on the optimal design of incentive schemes using a bilevel max-min optimization approach within a Stackelberg

game framework.

- *Agency problem, stochastic control and optimization.* Stochastic bilevel optimization in continuous time with ambiguity has been previously studied in Sung (2022); Mastrolia and Possamaï (2018); Hernández-Santibánez and Mastrolia (2019). In this work, we extend this framework to a stochastic bilevel max-min optimization problem in continuous time and volatility uncertainty with jumps. Specifically, we propose a novel connection between second-order backward stochastic differential equations with jumps (2BSDEJs) and principal-agent problems involving both moral hazard and model ambiguity. 2BSDEs have been extensively studied in the literature since the pioneering works Soner et al. (2012); Cheridito et al. (2007); Possamaï et al. (2018); see also Popier and Zhou (2019); Possamaï and Tan (2015); Matoussi et al. (2014), and more recently, their extensions to include jump processes Kazi-Tani et al. (2015); Denis et al. (2024). However, the link between 2BSDEs with jumps and principal-agent problems under volatility uncertainty and accident risk has not yet been established. This paper addresses that gap. In particular, we extend the framework of Hernández-Santibánez and Mastrolia (2019) to incorporate accidents, and generalize the models in Capponi and Frei (2015); Bensalem et al. (2020) by introducing volatility ambiguity in the context of cyber risk. Finally, we develop a Perron's method to prove the existence of a viscosity solution to an integro-partial Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation characterized by the principal's value function $V_0^P$. This extends the methods in Sîrbu (2014); Bayraktar and Sirbu (2012) and Hernández-Santibánez and Mastrolia (2019) to settings with jump-diffusion processes.

The structure of this work is as follows. Section 2 presents the modeling framework, including the canonical process and weak formulation of the problem, the controlled stochastic SIR and price models, L-hop risk propagation, admissible controls and contracts, and finally the bilevel max-min stochastic optimization. Section 3 focuses on the $(IC - \sigma)$ condition—also known as the agent's problem—and its connection to a 2BSDE with jumps. Section 4 investigates the optimal compensation schemes by reducing the problem to an integro-Isaacs PDE, applying a verification theorem and Perron's method in the context of discontinuous stochastic processes.

# 2 Cyber risk modeling and bilevel max-min problem

## 2.1 Canonical process and weak formulation

Fix a maturity $T > 0$ and positive integers $n, m$. Let $C([0,T], \mathbb{R}^n)$ be the space of continuous functions from $[0,T]$ to $\mathbb{R}^n$, and define

$$\Omega^c := \{\omega^c \in C([0,T], \mathbb{R}^n) : \omega_0^c = 0\}, \quad \Omega^d := \mathbb{D}([0,T]; \mathbb{R}^m),$$

where $\mathbb{D}([0,T]; \mathbb{R}^m)$ is the space of càdlàg path on $[0,T]$ with values in $\mathbb{R}^n$. We call $\Omega := \Omega^c \times \Omega^d$ the *canonical space*, equipped with the uniform norm $\|\omega\|_\infty := \sup_{t\in[0,T]} \|\omega_t\|$.

We denote by $X$ the *canonical process* on $\Omega$, i.e. for $\omega \in \Omega$ and $t \in [0,T]$, $X_t(\omega) := \omega_t$. Let $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ be the filtration generated by $X$, and let $\mathbb{G}^+ := (\mathcal{G}_t^+)_{t\in[0,T]}$ denote its right-continuous modification, where $\mathcal{G}_t^+ := \bigcap_{s>t} \mathcal{G}_s, \quad t \in [0,T)$, and $\mathcal{G}_T^+ := \mathcal{G}_T$. We denote by $\mathbb{P}_0$ the *Wiener measure* on $(\Omega, \mathcal{G}_T)$, and by $\mathcal{M}(\Omega)$ the set of all probability measures on $(\Omega, \mathcal{G}_T)$. Next, we recall *universal filtration* $\mathbb{G}^* = (\mathcal{G}_t^*)_{t\in[0,T]}$, defined by $\mathcal{G}_t^* := \bigcap_{\mathbb{P}\in\mathcal{M}(\Omega)} \mathcal{G}_t^{\mathbb{P}}$, where $\mathcal{G}_t^{\mathbb{P}}$ is the usual completion of $\mathcal{G}_t$ in $\mathbb{P}$. For any subset $\mathcal{P} \subset \mathcal{M}(\Omega)$, a set is said to be $\mathcal{P}$*-polar* if it is $\mathbb{P}$-negligible for every $\mathbb{P} \in \mathcal{P}$. A property is said to hold $\mathcal{P}$*-quasi-surely* if it fails only on a $\mathcal{P}$-polar set.

We then define the filtration $\mathbb{F}^{\mathcal{P}} := (\mathcal{F}_t^{\mathcal{P}})_{t\in[0,T]}$ by $\mathcal{F}_t^{\mathcal{P}} := \mathcal{G}_t^* \vee \mathcal{T}^{\mathcal{P}}, \quad t \in [0,T]$, where $\mathcal{T}^{\mathcal{P}}$ is the collection of all $\mathcal{P}$-polar sets. Its right-continuous modification is denoted by $\mathbb{F}^{\mathcal{P},+} := (\mathcal{F}_t^{\mathcal{P},+})_{t\in[0,T]}$. When there is no risk of confusion, we omit the index $\mathcal{P}$.

For any subset $\mathcal{P} \subset \mathcal{M}(\Omega)$ and any pair $(t, \mathbb{P})$ with $t \in [0,T]$ and $\mathbb{P} \in \mathcal{P}$, we define

$$\mathcal{P}[\mathbb{P}, \mathcal{F}_t, t] := \{\mathbb{P}' \in \mathcal{P} : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\}.$$

It is well known (see, e.g., Stroock and Varadhan (1997)) that for every $\mathbb{P} \in \mathcal{M}(\Omega)$ and every $\mathbb{F}$-stopping time $\tau$ with values in $[0,T]$, there exists a family of *regular conditional probability distributions (r.c.p.d.)* $(\mathbb{P}_\omega^\tau)_{\omega\in\Omega}$; we refer to (Possamaï et al., 2018, Section 1.1.3) for details.

We associate to the jumps of $X$ the *counting measure* $\mu_X$ on $E \times (0, \infty)$, with $E \in \mathcal{B}(\mathbb{R}^m \backslash \{0\})$, defined pathwise by

$$\mu_X(E, [0, t]) := \sum_{0<s\leq t} \mathbf{1}_{\{\Delta X_s \in E\}}, \quad \text{for } t \geq 0.$$

We assume the counting measure $\mu_X$ could always admit a decomposition at time $t$:

$$\mu_X(dx, dt) = \nu_t(dx)dt,$$

where $\nu_t(\cdot) := \mathbb{E}[\mu_X(\cdot, [t, t+1])]$ is a $\sigma$-finite measure on $\mathcal{B}(E)$, *i.e.* the Lévy measure.

Let $\mathcal{P}_S$ denote the set of all semimartingale measures $\mathbb{P} \in \mathcal{M}(\Omega)$ such that

(o) $\mathbb{P}$ can be represented by a decomposition of the measure $\mathbb{P} = \mathbb{P}^c \otimes \mathbb{P}^d$ followed from Lebesgue decomposition theorem, where $\mathbb{P}^c$ is absolute continuous with respect to $\mathbb{P}$ and $\mathbb{P}^d$ is the pure point part. Note that we omit to add a singular probability as it is not contributing to the canonical process $X$ and so can disapears under any expectation we are considering.

(i) $(X_s)_{s\in[t,T]}$ be a $(\mathbb{P}, \mathbb{F})-$semimartingale. By the canonical decomposition (see, e.g., (Jacod and Shiryaev, 2013, Theorem I.4.18)), for $s \in [t, T]$ and $\mathbb{P}$-almost surely,

$$X_s \;=\; \int_t^s b_r^{\mathbb{P}}\, dr \;+\; X_s^{c,\mathbb{P}} \;+\; X_s^{d,\mathbb{P}},$$

where $b^{\mathbb{P}}$ is an $\mathbb{F}^{\mathbb{P}}$-predictable, $\mathbb{R}^n$-valued process, $X^{c,\mathbb{P}}$ is the continuous local martingale part of $X$ (measured by $\mathbb{P}^c$), and $X^{d,\mathbb{P}}$ is the purely discontinuous local martingale part of $X$ (measured by $\mathbb{P}^d$).

(ii) The quadratic variation of $X^{c,\mathbb{P}}$ is absolutely continuous with respect to the Lebesgue measure $dt$. Its density takes values in $\mathbb{S}_n^{\geq 0}$, the space of all $n \times n$ real-valued positive semidefinite matrices.

(iii) The compensator $\lambda_t^{\mathbb{P}}(dx, dt)$ of the jump measure $\mu_X$ exists under $\mathbb{P}$ (or equivalently $\mathbb{P}^d$). Moreover, there is an $\mathbb{F}$-predictable random measure $\nu^{\mathbb{P}}$ on $E$ such that

$$\lambda_t^{\mathbb{P}}(dx, dt) \;=\; \nu_t^{\mathbb{P}}(dx)\, dt.$$

We will denote by $\tilde{\mu}^{X,\mathbb{P}}(dx, dt)$ the corresponding compensated measure.

It is well known (see, e.g., Karandikar (1995)) that there exists an $\mathbb{F}-$progressively measurable process $\langle X\rangle := (\langle X\rangle_t)_{t\in[0,T]}$, which coincides $\mathbb{P}-$a.s. with the quadratic variation of $X$ for each $\mathbb{P}\in\mathcal{P}_S$. Its density with respect to Lebesgue measure at any $t\in[0,T]$ is given by a nonnegative symmetric matrix $\widehat{\sigma}_t\in\mathcal{M}_{n,n}(\mathbb{R})$, defined by

$$\widehat{\sigma}_t := \limsup_{\varepsilon\to 0^+}\frac{\langle X^c\rangle_t - \langle X^c\rangle_{t-\varepsilon}}{\varepsilon},$$

where $\langle X^c\rangle$ is the pathwise continuous part of $\langle X\rangle$.

We refer to Appendix A for the formal definitions of functional spaces mentioned hereafter.

Let $A, H$ be compact subsets of some finite-dimensional spaces. We define

$$\mathfrak{A} \ := \ \Big\{\alpha :\ [0,T]\times\Omega\to A \text{ is } \mathbb{F}\text{−adapted}\Big\}, \qquad \mathfrak{H} \ := \ \Big\{\eta :\ [0,T]\times\Omega\to H \text{ is } \mathbb{F}\text{−adapted}\Big\},$$

where $A\subset\mathbb{R}^{k_1},\ H\subset\mathbb{R}^{k_2}$ for some integers $k_1,k_2$. The process $\alpha$ represents the control of the *Subsidiary* (the agent), while $\eta$ is the adversarial control of the *Hacker*.

Next, consider the volatility coefficient

$$\sigma :\ [0,T]\times\Omega\times H \ \longrightarrow\ \mathcal{M}_{n,\ell}(\mathbb{R}),$$

where $\mathcal{M}_{n,\ell}(\mathbb{R})$ denotes the space of real $n\times\ell$ matrices. We assume that $\sigma$ is an $\mathbb{F}$-progressively measurable uniformly bounded process such that $\sigma\,\sigma^\top(\cdot,\mathfrak{h})$ is invertible for every $\mathfrak{h}\in H$. For each pair $(t,x)\in[0,T]\times\Omega$ and $\eta\in\mathfrak{H}$, we define an $\ell$-dimensional Brownian motion $W$ and an $m-$dimensional mark process $\mu_X^0$ with predictable intensity kernel $\lambda_s^0$ so that for $E\in\mathcal{B}_0$, $\mu_X^0(\mathcal{E},[t,s]) = \tilde{\mu}_X^0(\mathcal{E},[t,s]) + \int_t^s\int_E L_s(X_{s-},e)\nu_s(de)ds$, where $\nu_s(\mathcal{E}) = \mathbb{E}[\mu_X^0(\mathcal{E},[s,s+1])$ and $L_r:\mathbb{R}^n\times E\longrightarrow\mathcal{M}_{n,m}(\mathbb{R})$. We consider the SDE driven by this Brownian and Poisson measure with solution denoted by $X$ and defined for any $s\in[t,T]$ by

$$X_s^{t,x,\eta} \ = \ x(t)+\int_t^s b^\eta(r,X_r^{t,x,\eta},\eta_r)dr+\int_t^s\sigma\big(r,X_r^{t,x,\eta},\eta_r\big)\,dW_r+\int_t^s\int_E L_r(X_{r-},e)\tilde{\mu}^0_{X_r^{t,x,\eta}}(de,dr), \tag{2.1}$$

$$X_r^{t,x,\eta} \;=\; x(r), \quad r \in [0,t],$$

where $b_r$ is an $\mathbb{R}^n$-valued Lipschitz function of $X$. From now on we only consider the probability measure that can be represented by $\mathbb{P} = \mathbb{P}^c \otimes \tilde{\mu}_X^0$.

**Weak formulation induced by cyber risk.** We build a control model via the *weak solutions* of (2.1). Specifically, a pair $(\mathbb{P}, \eta)$ is called a *weak solution* of (2.1) if the law of $X_t^{t,x,\eta}$ under $\mathbb{P}^c$ is $\delta_{x(t)}$, and there exists a $\mathbb{P}^c$-Brownian motion, denoted $W^{\mathbb{P}^c}$, and a Poisson measure $\mu_{X_r^{t,x;\eta}}$ with predictable intensity kernel $\lambda(X_s, \eta_s)$ and corresponding martingale $\tilde{\mu}_{X_r^{t,x,\eta}}$ such that

$$X_s \;=\; x(t) + \int_t^s b^\eta(r, X_r^{t,x,\eta}, \eta_r)dr + \int_t^s \sigma\big(r, X_r^{t,x,\eta}, \eta_r\big)\, dW_r^{\mathbb{P}^c} + \int_t^s \int_E L_r(X_{r-}, e)\tilde{\mu}_{X_r^{t,x,\eta}}(de, dr),$$

for any $t \le s \le T$, $\quad \mathbb{P}$-a.s.

Let $\widetilde{\mathcal{H}}(t,x)$ denote the set of all weak solutions $(\mathbb{P}, \eta)$ to the SDE (2.1). Note that this set is not empty by Girsanov theorem for jump processes, see for instance (Papapantoleon, 2008, Section 12) with

$$\tilde{\mu}_X(de, dt) := \mu^0(de, dt) - \lambda(X_{t-}, \eta_t)\nu_t(de)dt.$$

We then define

$$\mathcal{P}(t,x) \;:=\; \bigcup_{\eta \in \mathfrak{H}} \mathcal{P}^\eta(t,x), \quad \text{where} \quad \mathcal{P}^\eta(t,x) \;:=\; \Big\{\mathbb{P} \in \mathcal{M}(\Omega) \,:\, (\mathbb{P}, \eta) \in \widetilde{\mathcal{H}}(t,x)\Big\}.$$

**Remark 1.** *The set $\mathcal{P}(t,x)$ will play a crucial role in ensuring the well-posedness of 2BSDEs, our main tool for solving the subsidiary's optimization problem. In order to use 2BSDEs effectively, we require $\mathcal{P}(t,x)$ to be* saturated. *Recall that a set $\mathcal{P} \subset \mathcal{M}(\Omega)$ is saturated if for any $\mathbb{P} \in \mathcal{P}$ and any probability measure $\mathbb{Q} \in \mathcal{M}(\Omega)$ that is equivalent to $\mathbb{P}$ (and under which $X$ is a local martingale), we have $\mathbb{Q} \in \mathcal{P}$. Following the same arguments as in (Cvitanić et al., 2018, Proof of Proposition 5.3, step (i)), we deduce that for each $(t,x) \in [0,T] \times \Omega$, the set $\mathcal{P}(t,x)$ is saturated.*

**Remark 2.** *In the classical framework, as in Mastrolia and Possamaï (2018); Hernández-Santibánez and Mastrolia (2019), the Principal and Agent may hold different beliefs about the volatility, leading to distinct sets of weak solutions to the SDE (2.1). However, in our problem setup, particularly in the context of a holding company and its subsidiary, it is customary to assume that they share the same belief.*

## 2.2 Cyber risk modeling: Controlled SIR-price system

We now turn to the particular cyber risk model we are considering by specifying the dynamic of $X$ with a controlled SIR model and the subsidiary's portfolio evolution. We model the computers or electronic devices in the cluster by SIR model, following the construction in Hillairet et al. (2024):

- **Susceptible (S)**: $S_t$ denotes the proportion of computers at time $t$ that are insufficiently protected and not yet infected, making them susceptible to attacks.
- **Infected (I)**: $I_t$ represents the proportion of infected and corrupted computers at time $t$ that can potentially contaminate other devices through cyber contagion and interconnectedness.
- **Recovery (R)**: $R_t$ indicates the proportion of computers at time $t$ that have either recovered from infection or are protected by antivirus software, rendering them immune to future infections.

The process $(S_t, I_t, R_t)_{t\geq 0}$ denotes the proportions of computers in the corresponding classes relative to the total number of computers. At each time $t$, the system must satisfy the condition:

$$S_t + I_t + R_t = 1.$$

**Notation and Interpretation:** We denote by $\beta > 0$ the transmission rate of an infected computer, defined as the average number of contacts per unit time made by an infected device that result in further infections. The hacker's action at time $t$, denoted by $\eta_t$, affects both the transmission rate and its volatility. Additionally, $\eta_t$ influences the volatility of the stock price. The number of infected computers also impacts both the expected return and the volatility of the stock price, denoted by $P$. The subsidiary's control strategy, denoted $\alpha_t$, is implemented to protect suceptible and uninfected computers from infection.

**Assumption 2.1** (Cyber risk canonical model)**.** *We consider the following cyber risk modeling setting, as a particular case of the model introduced in Section 2.1.*

- *The canonical process is decomposed into three canonical variables* $\mathbf{X}_s := (P_s, S_s, I_s)^\top$, *so that* $n = 3$ *in the setting of the previous section.*
- *The jumps induced by the cyber attack only affect the portfolio value process* $P$. *The matrix* $L_r$ *is such that* $(L_r(\mathbf{X}_{r-}, e))^{j,k} = 0$ *for any* $j > 1$ *and* $k \leq m$. *For the sake of notation simplicity, we then denotes by* $\mu_P$ *the measure* $\mu_X$ *introduced previously since it is affecting only the process* $P$.

We define the drift and volatility process under cyber threat and hacking activity by

$$\mathbf{b}^\eta(s, \mathbf{X}_s, \eta_s) := \begin{pmatrix} \mu(s, I_s)P_s \\ -\beta S_s I_s - \eta_s S_s \\ \beta S_s I_s + \eta_s S_s - \nu I_s \end{pmatrix}, \quad \boldsymbol{\sigma}(s, \mathbf{X}_s, \eta_s) = \begin{pmatrix} \sigma(s, \eta_s)P_s & 0 \\ 0 & -\tilde{\sigma}(s, \eta_s) I_s S_s \\ 0 & +\tilde{\sigma}(s, \eta_s) I_s S_s \end{pmatrix}$$

and the dynamic of $\mathbf{X}$ for any $(\mathbb{P}, \eta) \in \widetilde{\mathcal{H}}(t, x)$ is given by

$$d\mathbf{X}_s = \mathbf{b}^\eta(s, \mathbf{X}_s, \eta_s)ds + \boldsymbol{\sigma}(s, \mathbf{X}_s, \eta_s)d\mathbf{W}_s + \int_E L_r(\mathbf{X}_{r-}, e)\tilde{\mu}_P(ds, de) \cdot P_s,$$

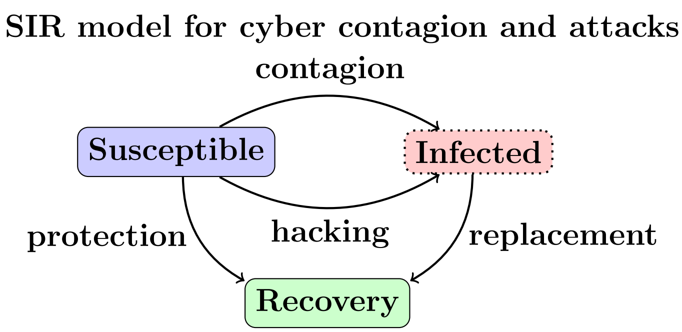
where $\mathbf{W}_s := (W_s, \widetilde{W}_s)$, is a two-dimensional Brownian motion under $\mathbb{P} \in \mathcal{P}_0$. From now on $\ell = 2$. Moreover, we assume that there exists a function $\lambda : [0, T] \times \mathbb{R} \times H \to \mathbb{R}^m$ that describes $\nu_s(\mathcal{E})$ as: $\nu_s(\mathcal{E}) = \lambda_s(I_s, \eta_s)$.

**Admissible Hacker attacks.** Note that the SIR process must admit global positive solution. From now on, we define $\mathcal{H}(t, x)$ as the set of admissible hacker control $(\mathbb{P}, \eta) \in \widetilde{\mathcal{H}}(t, x)$ such that $(\mathbf{X}^2, \mathbf{X}^3) = (S, I)$ is a pair of positive processes with values in $[0, 1]$. This set is not empty by considering for example constant hacking activities $\eta_t = \eta > 0$.

## 2.3 Admissible efforts of the subsidiary

In this section we elaborate on the admissible conditions of the control and the impact of the actions to the outcome process. The subsidiary company (the agent) exerts an effort $\alpha \in \mathfrak{A}$ to manage the whole system, which is unobservable by the holding company (the principal), impacting the outcome process through the drift coefficient $\mathbf{b} : [0,T] \times \mathbb{R}^n \times \Omega \times A \times H \to \mathbb{R}^n$, which satisfies that $b(\cdot, \mathfrak{a}, \mathfrak{h})$ is an $\mathbb{F}$-progressively measurable process for every $(\mathfrak{a}, \mathfrak{h}) \in A \times H$.

We first introduce the informal definition of the controlled SIR system and the portfolio value. We model the cyber-attack in a short time period as the following SIR dynamics similar as in Hillairet et al. (2022) and Hillairet et al. (2024):




The corresponding stochastic SIR systems evolves as follow

$$\begin{cases} dS_t = (-\beta S_t I_t - \alpha_t S_t - \eta_t S_t)dt - \tilde{\sigma}(s,\eta_t) I_t S_t d\widetilde{W}_t \\ dI_t = (\beta S_t I_t - \rho I_t + \eta_t S_t)dt + \tilde{\sigma}(s,\eta_t) I_t S_t d\widetilde{W}_t \\ dR_t = \rho I_t dt + \alpha_t S_t dt, \end{cases} \tag{2.2}$$

while the subsidiary value is given by

$$dP_s = P_s \left( \mu(s, I_s)dt + \sigma(s, I_s, \eta_s)dW_s + \int_E L_s^{1,\cdot}(\mathbf{X}_{s-}, e)\mu_P(de, ds) \right),$$

where the drift $\mu : [0,T] \times [0,1] \longrightarrow \mathbb{R}$ and $\sigma : [0,T] \times [0,1] \times H \longrightarrow \mathbb{R}^+$ are assumed to be bounded.

We now provide a rigorous probabilistic formulation of this controlled model. We define

$$\boldsymbol{\beta}(\mathbf{X}_s;\alpha_s) := \begin{pmatrix} 0 \\ -\alpha_s S_s \\ 0, \end{pmatrix} \quad \mathbf{b}^d(s,\mathbf{X}_s;\eta_s) := \begin{pmatrix} \int_E L_s(\mathbf{X}_{s-},e)\nu_s(de)P_s \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{b}^\alpha(s,\mathbf{X}_s;\alpha_s,\eta_s) = \boldsymbol{\beta}(\mathbf{X}_s;\alpha_s) + \mathbf{b}^d(s,\mathbf{X}_s;\eta_s),$$

and the drift process of $\mathbf{X}$ and its continuous part

$$\mathbf{b}(s,\mathbf{X}_s;\alpha_s,\eta_s) = \mathbf{b}^\alpha(s,\mathbf{X}_s;\alpha_s,\eta_s)+\mathbf{b}^\eta(s,\mathbf{X}_s,\eta_s), \quad \mathbf{b}^c(s,\mathbf{X}_s;\alpha_s,\eta_s) = \boldsymbol{\beta}(\mathbf{X}_s;\alpha_s)+\mathbf{b}^\eta(s,\mathbf{X}_s,\eta_s).$$

**Remark 3.** *We note that the drift $\boldsymbol{b}$ satisfies*

$$\left\|\boldsymbol{b}(t,\mathbf{x},\mathfrak{a},\mathfrak{h})\right\| \;\le\; \kappa\Big(1+\|x\|_{t,\infty}+|\mathfrak{a}|\Big), \quad \left\|\partial_{\mathfrak{a}}\, b(t,\mathbf{x},\mathfrak{a},\mathfrak{h})\right\| \;\le\; \kappa\Big).$$

**Definition 2.1** (Admissible control)**.** *A control process $\alpha = (\alpha_t)_{t\in[0,T]}$ is said to be* admissible *if for every $(\mathbb{P},\eta)\in\mathcal{H}(t,x)$, the following Doleans Dade exponential process,*

$$\mathcal{E}_t^{\alpha,\eta} := \exp\Bigg(\int_0^t \boldsymbol{\sigma}^\top(\boldsymbol{\sigma}\boldsymbol{\sigma}^\top)^{-1}(s,\mathbf{X}_s,\eta_s)\boldsymbol{\beta}(\mathbf{X}_s;\alpha_s)\cdot dW_s^{\mathbb{P}} - \frac{1}{2}\int_0^t \left\|\boldsymbol{\sigma}^\top(\boldsymbol{\sigma}\boldsymbol{\sigma}^\top)^{-1}(s,\mathbf{X}_s,\eta_s)\boldsymbol{\beta}(\mathbf{X}_s;\alpha_s)\right\|^2 ds\Bigg), \tag{2.3}$$

*is a $(\mathbb{F},\mathbb{P})$-martingale and if (2.2) admits a unique strong solution with $S_t, I_t, R_t$ takes values in $\mathbb{R}_+$ and $S_t+I_t+R_t=1$. We denote by $\mathcal{A}$ the set of all such admissible controls.*

**Remark 4.** *The set $\mathcal{A}$ is not empty since it contains the control $\alpha$ with the form $\alpha_t = \alpha I_t S_t$ with $\alpha$ a nonnegative constant. This example ensures that (2.3) is a martingale and (2.2) has a positive global solution, see for example (Jiang et al., 2011, Theorem 2.1).*

We can then define the impact of the control $\alpha\in\mathcal{A}$ by changing the distribution of the system through the following set of probability measures

$$\mathcal{P}^\alpha \;=\; \left\{(\mathbb{P}^{\alpha,\eta},\eta), \frac{d\mathbb{P}^{\alpha,\eta}}{d\mathbb{P}} = \mathcal{E}_T^{\alpha,\eta},\ (\mathbb{P},\eta)\in\mathcal{H}\right\}.$$

Then, by Girsanov's Theorem, for any $\alpha \in \mathcal{A}$ and $(\mathbb{P}^{\alpha,\eta}, \eta) \in \mathcal{P}^{\alpha}$, we have

$$\mathbf{X}_s = x_t + \int_t^s \mathbf{b}(r, \mathbf{X}_r; \alpha_r, \eta_r)dr + \int_t^s \boldsymbol{\sigma}(r, \mathbf{X}_r, \eta_r)d\mathbf{W}_r^{\eta} + \int_t^s L_r(\mathbf{X}_{r-}, e)\tilde{\mu}_P(dr, de) \cdot P_s,$$
$$s \in [t, T],\ \mathbb{P}^{\alpha,\eta} - a.s.,$$

where $\mathbf{W}^{\eta}$ is a $\mathbb{P}^{\alpha,\eta}$-Brownian motion and $\tilde{\mu}_P(dr, de) := \mu_P^0(de, ds) - \lambda_s(I_s, \eta_s)\nu_E(de)ds$ is compensated jump measure under $\mathbb{P}^{\alpha,\eta}$.

## 2.4 L-hop modeling

Assume that $\boldsymbol{\lambda}(I_s, \eta_s) = (\lambda^{e,1}(\eta_s), \ldots, \lambda^{e,m^e}(\eta_s), \lambda^{i,1}(I_s), \ldots, \lambda^{i,m^i}(I_s))^{\top}$ where $m = m^e + m^i$. In this example, the jump in price are driven by two forces: the attacker outside the system and the internal effect of the infected computer. This illustrates the L-hop risk propagation with external and internal attacks and risk. An even more specific example when $m = 2$ would be to consider

$$L_r(\mathbf{X}_{r-}, e) = \begin{pmatrix} -c^e & -c^i \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda(I_s, \eta_s) = \begin{pmatrix} \lambda^e(\eta_s) \\ \lambda^i(I_s) \end{pmatrix},$$

so that the portfolio is impacted by the L-hop attacks

$$dP_s = P_s \left( \mu(s, I_s)dt + \sigma(s, I_s, \eta_s)dW_s - c^e dN_s^e - c^i dN_s^i \right),$$

where $N = (N^e, N^i)$ is a $2-$dimensional Poisson process with intensity $\lambda(I_s, \eta_s)$.

## 2.5 Bilevel max-min cyber risk optimization

The holding company offers to the subsidiary an $\mathcal{F}_T-$measurable compensation $\xi$ at time $T$. The subsidiary benefits form both the compensation and the portfolio value at time $T$ with utilities $U^A$ and $F^A$, where $U_A$ is assumed to be concave while $F^A$ has a polynomial growth in $\mathbf{X}$. The monitoring activities of the subsidiary to reduce the cyber attack propagation within the SIR cluster with control process $\alpha$ induced a cost $f : [0, T] \times \mathbb{R}^d \times A \longrightarrow \mathbb{R}$

while the attacks induced costs modeled through a marked process $N^A$ with compensator $\tilde{N}^A(ds,de) = N^A(ds,de) - \lambda^A(\mathbf{X}_s)\nu^A(de)ds$ on $E^A \in \mathcal{B}(\mathbb{R}\setminus\{0\})$ for some intensity function $\lambda^A : \mathbb{R}^3 \longrightarrow \mathbb{R}^+$ with cost $L^A : [0,T]\times\mathbb{R}^3\times E^A \longrightarrow \mathbb{R}$.

Given a compensation scheme $\xi$, the problem of the subsidiary is

$$V_0^A(\xi) = \sup_{\alpha\in\mathcal{A}} \inf_{(\mathbb{P},\eta)\in\mathcal{P}^\alpha} \mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{0,T}\Big(U^A(\xi) + F^A(\mathbf{X}_T)\Big) - \int_0^T \mathcal{K}_{0,s}C^A(s,\mathbf{X}_s,\alpha_s)ds\right],$$

where $\mathcal{K}_{t,s} := \exp\left(\int_t^s k(r,\mathbf{X}_r)dr\right)$, for any $0\le t\le s\le T$, and $C^A : [0,T]\times\mathbb{R}^3\times A \longrightarrow \mathbb{R}$ defined by

$$C^A(s,x,\mathfrak{a}) = f(s,x,\mathfrak{a}) + \int_{E^A} L_s^A(x,e)\lambda^a(x)\nu^A(de).$$

The holding firm benefits from the portfolio value of the subsidiary and its cyber sanity through a payoff function $F^P : \mathbb{R}^3 \longrightarrow \mathbb{R}$. We also assume that the holding firm aims to reduce the variability of cyber risk propagation in the SIR-cluster system with a quadratic penalty term $\varepsilon > 0$ and is also subject to costs induced by the consequences of the cyber attack modeled through a marked process $N^P$ with compensator $\tilde{N}^P(ds,de) = N^P(ds,de) - \lambda^P(\mathbf{X}_s)\nu^P(de)ds$ on $E^P \in \mathcal{B}(\mathbb{R}\setminus\{0\})$ for some intensity function $\lambda^P : \mathbb{R}^3 \longrightarrow \mathbb{R}^+$ with cost $L^P : [0,T]\times\mathbb{R}^3\times E^P \longrightarrow \mathbb{R}$. We denote $C^P : [0,T]\times\mathbb{R}^3\times H \longrightarrow \mathbb{R}$ the cost of cyber attacks' consequences and uncertainty and define for any $x=(p,s,i)$ and $\mathfrak{h}\in H$ by

$$C^P(s,x,\mathfrak{h}) = \frac{1}{2}\varepsilon|\tilde{\sigma}(s,\mathfrak{h})si|^2 + \int_{E^P} \lambda_s^P(x)L_s(x,e)\nu^P(de).$$

In this adversarial scenario, the agency problem is reduced to solve the following max-min bilevel optimization under constraint

**(2Mm-$\sigma$)** $\quad V_0^P := \sup_{\xi\in\Xi,\hat{\alpha}(\xi)\in\mathcal{A}} \inf_{(\mathbb{P},\eta)\in\mathcal{P}^{\hat{\alpha}(\xi)}} \mathbb{E}^{\mathbb{P}}\Big[F^P(\mathbf{X}_T) - \xi - \int_0^T C^P(s,\mathbf{X}_s,\eta_s)ds\Big]$

subject to

$$(IC-\sigma):\quad V_0^A(\xi) = \inf_{(\mathbb{P},\eta)\in\mathcal{P}^{\hat{\alpha}(\xi)}} \mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{0,T}\Big(U^A(\xi)+F^A(\mathbf{X}_T)\Big) - \int_0^T \mathcal{K}_{0,s}C^A(s,\mathbf{X}_s,\hat{\alpha}_s(\xi))ds\right]$$

$$(R):\quad V_0^A(\xi) \ge R_0.$$

# 3 Subsidiary company's problem: 2BSDE with jumps and dynamic programming

In this section, we aim at solving the condition $(IC-\sigma)$ that is to find for any $\xi \in \Xi$ a best reaction strategy of the subsidary $\hat{\alpha}$ under the worst-cyber attack scenario $(\hat{\mathbb{P}}, \hat{\eta})$

$$V_0^A(\xi) = \sup_{\alpha\in\mathcal{A}} \inf_{(\mathbb{P},\eta)\in\mathcal{P}^\alpha} \mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{0,T}\Big(U^A(\xi) + F^A(\mathbf{X}_T)\Big) - \int_0^T \mathcal{K}_{0,s} C^A(s, \mathbf{X}_s, \alpha_s) ds\right].$$

## 3.1 Definition of the Hamiltonian

Define the function $G : [0,T] \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^4 \times \mathcal{L}_\nu^{p,m} \times A \times H \to \mathbb{R}$ by

$$G(t, \mathbf{x}, y, z, u, \mathfrak{a}, \mathfrak{h}) := -k(t,\mathbf{x})y - C^A(t, \mathbf{x}, \mathfrak{a}) + \mathbf{b}^c(t, \mathbf{x}; \mathfrak{a}, \mathfrak{h})\cdot z + \int_E u_s(e)\cdot\{\lambda(\mathbf{x}^3, \mathfrak{h}) - \lambda_t^0\}\nu_t(de)\mathbf{x}^1,$$

recalling that

$$C^A(t, \mathbf{x}, \mathfrak{a}) := f(t, \mathbf{x}, \mathfrak{a}) + \int_{E^A} L_t^A(\mathbf{x}, e)\, \lambda^{\mathfrak{a}}(\mathbf{x})\, \nu^A(de).$$

Define also for every $(t, \mathbf{x}, \Sigma) \in [0,T] \times \Omega \times \mathcal{S}_3^+$ the set

$$\mathcal{H}_t(\mathbf{x}, \Sigma) := \left\{\eta \in \mathcal{H},\ \boldsymbol{\sigma}(t, \mathbf{x}, \eta_t)\boldsymbol{\sigma}^\top(t, \mathbf{x}, \eta_t) = \Sigma\right\},$$

and denote by $\mathcal{H}(\hat{\sigma}^2)$ the set of controls $\eta \in \mathcal{H}$ with values in $\mathcal{H}_t(\mathbf{x}, \hat{\sigma}_t^2)$, $dt \otimes \mathbb{P}$-a.e. for every $\mathbb{P} \in \mathcal{P}$.

The Hamiltonian $\widehat{G} : [0,T] \times \mathbb{R}^4 \times \Omega \times \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^m \times S_3^+ \to \mathbb{R}$ associated with the problem of the Agent is defined by

$$\widehat{G}(t, \mathbf{x}, y, z, u, \gamma) := \inf_{\Sigma\in S_4^+}\left\{\frac{1}{2}\mathrm{Tr}(\Sigma\gamma) + \inf_{\eta\in\mathcal{H}_t(x,\Sigma)} \sup_{\alpha\in\mathcal{A}} G(t, \mathbf{x}, y, z, u, \alpha, \eta)\right\}.$$

**Assumption 3.1** (Isaacs condition)**.** *The following Isaacs' condition is satisfied for any* $(t, \mathbf{x}, \mathfrak{d}, y, z, u, \Sigma) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^m \times S_4^+$ *:*

$$\inf_{\eta\in\mathcal{H}_t(x,\Sigma)} \sup_{\alpha\in\mathcal{A}} G\big(t, \mathbf{x}, y, z, u, \alpha, \eta\big) \ = \ \sup_{\alpha\in\mathcal{A}} \inf_{\eta\in\mathcal{H}_t(x,\Sigma)} G\big(t, \mathbf{x}, y, z, u, \alpha, \eta\big). \tag{3.1}$$

Define the map $G^* : [0,T] \times \mathbb{R}^4 \times \Omega \times \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^m \times \mathcal{S}_4^+ \to \mathbb{R}$

$$G^*(t, \mathbf{x}, y, z, u, \Sigma) := \sup_{\alpha \in \mathcal{A}} \inf_{\eta \in \mathcal{H}_t(x,\Sigma)} G(t, \mathbf{x}, y, z, \alpha, \eta)$$

We now refer to a fundamental lemma on the growth of any control $\alpha^\star$ which is a saddle point in the Isaacs' condition (3.1).

**Assumption 3.2** (**H**). *There exists $\kappa > 0$ such that for any $(t, \mathbf{x}, \mathfrak{d}, \mathfrak{a}, \mathfrak{h}) \in [0,T] \times \mathbb{R}^4 \times \Omega \times A \times H$,*

(i) *the map $\mathfrak{a} \longmapsto f(t, \mathbf{x}, \mathfrak{a})$ is convex and continuously differentiable such that*

$$0 \;\leq\; f(t, \mathbf{x}, \mathfrak{a}) \;\leq\; \kappa\Big(1 + \|x\|_{t,\infty} + |\mathfrak{a}|^2\Big),$$

$$\big|\partial_{\mathfrak{a}} f(t, \mathbf{x}, \mathfrak{a})\big| \;\leq\; \kappa\Big(1 + |\mathfrak{a}|\Big), \quad \textit{and} \lim_{\|\mathfrak{a}\| \to \infty} \frac{f(t, \mathbf{x}, \mathfrak{a})}{|\mathfrak{a}|} \;=\; +\infty.$$

(ii) *The discount factor $k$ is uniformly bounded by $\kappa$.*

**Lemma 3.1.** *Let Assumption $\big(H\big)$ hold. Then, for any $(t, \mathbf{x}, y, z, u, \Sigma) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^m \times S_3^+$ the map $(y, z, u) \longrightarrow G^\star(t, \mathbf{x}, y, z, u, \Sigma)$ is Lipschitz.*

This lemma is a direct consequence of the compactness property of $A$ and $H$.

## 3.2 2BSDEJ representation of subsidiary's Problem

Consider the following 2BSDEJ

$$\begin{aligned} Y_t = U_A(\xi) + F^A(\mathbf{X}_T) + \int_t^T G^*(s, \mathbf{X}_s, Y_s, Z_s, U_s, \hat{\sigma}_s^2) ds \\ - \int_t^T Z_s \cdot d\mathbf{X}_s^c - \int_t^T dK_s - \int_t^T \int_E U_s(e) \tilde{\mu}_P^0(ds, de), \quad \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_0. \end{aligned} \tag{3.2}$$

**Definition 3.1.** *We say that a quadruplet* $(Y, Z, U, K)$ *is a solution to the 2BSDEJ* (3.2) *if there exists* $p > 1$ *such that*

$$(Y, Z, U, K) \in \mathbb{S}_0^p(\mathbb{F}_+^{\mathcal{P}}, \mathcal{P}) \times \mathbb{H}_0^p(\mathbb{F}^{\mathcal{P}}, \mathcal{P}) \times \mathbb{J}_0^p(\mathbb{F}^{\mathcal{P}}, \mathcal{P}) \times \mathbb{K}_0^p(\mathbb{F}^{\mathcal{P}}, \mathcal{P})$$

*satisfies* (3.2) *and* $K$ *satisfies the minimality condition*

$$0 = \underset{\mathbb{P}' \in \mathcal{P}[\mathbb{P}, \mathbb{F}+, s]}{\operatorname{essinf}} \mathbb{E}^{\mathbb{P}'} \left[ K_T - K_s \mid \mathcal{F}_s^{\mathbb{P},+} \right], \quad s \in [t, T],\ \mathbb{P}\text{ - } a.s.,\ \forall \mathbb{P} \in \mathcal{P}.$$

**Remark 5.** *In a more rigorous formulation, a solution should be given by the quadruple* $(Y, Z, U^{\mathbb{P}}, K^{\mathbb{P}})$*; see Remark 2.2 in Denis et al. (2024). Note that the process* $Z$ *can be aggregated and does not depend on* $\mathbb{P}$*, by applying similar results from Nutz and van Handel (2013), under additional technical assumptions on the set of considered probability measures. However, the aggregation of the process* $U$ *is not automatic, since the random measure associated with the corresponding integral depends on* $\mathbb{P}$*, as does its compensator. This poses a challenge in Principal-Agent problems where the probability measure is controlled by the agent. We will see in the next section that the Principal indexes the agent's compensation based on the process* $U$*. If this control variable depends on the probability measure influenced by the agent's actions, the contracting problem becomes ill-posed.*

*Nevertheless, in our specific framework, the probability measure* $\mathbb{P}$*, such that* $(\mathbb{P}, \eta) \in \mathcal{H}(t, x)$*, depends only on the external hacking variable* $\eta$*, and not on the agent's decisions. Thanks to the distinction between* $\mathcal{P}$ *and* $\mathcal{P}^{\alpha}$ *in Section 2, we can omit the dependence of* $U$ *on* $\mathbb{P}$*.*

We have the following result, which ensures that the 2BSDEJ (3.2) is well-posed as a consequence of Lemma 3.1 above and (Denis et al., 2024, Theorem 2.1 and Lemma 2.11) extending to multidimensional jump process.

**Lemma 3.2.** *Under Assumption (S), the 2BSDEJ* (3.2) *has a unique solution* $\big(Y, Z, U, K\big)$ *for any* $\mathcal{F}_T-$*measurable random variable* $\xi$ *such that* $U_A(\xi) \in \mathbb{L}_0^{p,\kappa}$ *.*

We now turn to the main result solving $(IC - \sigma)$.

**Theorem 3.3.** *Let $(Y, Z, U, K)$ be the solution to the 2BSDEJ (3.2). Then, the value function of the subsidiary solving $(IC-\sigma)$ is given by*

$$V_0^A(\xi) = \sup_{\alpha \in \mathcal{A}} \inf_{(\mathbb{P},\eta)\in\mathcal{P}^\alpha} \mathbb{E}^{\mathbb{P}}[Y_0]. \tag{3.3}$$

*Moreover, $(\hat{\alpha}; \mathbb{P}^\star, \eta^\star)$ is optimal for $(IC-\sigma)$ if and only if $(\hat{\alpha}; \mathbb{P}^\star, \eta^\star) \in \mathcal{A} \times \mathcal{P}$ and satisfies:*

*(i) $(\hat{\alpha}, \eta^\star)$ attains the sup-inf in the definition of $G^\star(\cdot, \mathbf{X}, Y, Z, U, \hat{\sigma}^2)$, $dt \otimes \mathbb{P}^\star$-a.e.,*

*(ii) $K_T = 0$, $\mathbb{P}^\star$-a.s.*

*Proof.* We follow the scheme in Hernández-Santibánez and Mastrolia (2019). We first prove that (3.3) holds with a characterization of the optimal effort of the Agent as a maximizer of the 2BSDEJ (3.2). The proof is divided into five steps.

**Step 1: BSDEJ and 2BSDJ.** For every $\big(\alpha, \eta\big) \in \mathcal{A} \times \mathcal{H}(\hat{\sigma}^2)$, denote by $\big(Y^{\alpha,\eta}, Z^{\alpha,\eta}, U^{\alpha,\eta}, K^{\alpha,\eta}\big)$ the solution of the following controlled 2BSDEJ in the sense of Definition 3.1 and where the wellposedness is deduced from Denis et al. (2024).

$$\begin{aligned} Y_t^{\alpha,\eta} &= U_A(\xi) + F^A(\mathbf{X}_T) \;+\; \int_t^T G\big(s, \mathbf{X}_s, Y_s^{\alpha,\eta}, Z_s^{\alpha,\eta}, U_s^{\alpha,\eta}, \alpha_s, \eta_s\big)\, ds \;-\; \int_t^T Z_s^{\alpha,\eta}\, d\mathbf{X}_s^c \\ &\quad - \int_t^T \int_E U_s^{\alpha,\eta}(e) \cdot \tilde{\mu}_P^0(ds, de) - \int_t^T dK_s^{\alpha,\eta}. \end{aligned}$$

Note in particular, see (Denis et al., 2024, Section 2.5) and (Possamaï et al., 2018, Theorem 4.2) that

$$Y_0^{\alpha,\eta} \;=\; \underset{\mathbb{P}'\in\mathcal{P}[\mathbb{P},\mathbb{F}^+,0]}{\operatorname{ess\,inf}}{}^{\mathbb{P}}\, \mathcal{Y}_0^{\mathbb{P}',\alpha,\eta}, \quad \mathbb{P}\text{-a.s. for every } \mathbb{P} \in \mathcal{P}. \tag{3.4}$$

where for any $\mathbb{P} \in \mathcal{P}$ the tuple $\big(\mathcal{Y}_t^{\mathbb{P},u,\alpha},\ \mathcal{Z}_t^{\mathbb{P},u,\alpha},\ \mathcal{U}_t^{\mathbb{P},u,\alpha}\big)$ is the solution of the following (well-posed) linear BSDEJ, see for example Papapantoleon et al. (2018)

$$\begin{aligned} \mathcal{Y}_t^{\mathbb{P},\alpha,\eta} \;&=\; U_A(\xi) + F^A(\mathbf{X}_T) \;+\; \int_t^T G\big(s, \mathbf{X}_s, \mathcal{Y}_s^{\mathbb{P},\alpha,\eta}, \mathcal{Z}_s^{\mathbb{P},\alpha,\eta}, \mathcal{U}_s^{\mathbb{P},\alpha,\eta}, \alpha_s, \eta_s\big)\, ds \;-\; \int_0^T \mathcal{Z}_s^{\mathbb{P},\alpha,\eta}\, d\mathbf{X}_s^c \\ &\quad - \int_0^T \int_E \mathcal{U}_s^{\mathbb{P},\alpha,\eta}(e) \cdot \tilde{\mu}_P^0(ds, de), \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.5}$$

Similarly, consider also, for every $\alpha \in \mathcal{A}$, the solution $\big(Y^\alpha, Z^\alpha, U^\alpha, K^\alpha\big)$ of the following 2BSDEJ, defined $\mathcal{P}$–q.s. by

$$Y_t^\alpha = U_A(\xi) + F^A(\mathbf{X}_T) + \int_t^T \inf_{\eta\in\mathcal{H}(x,\hat{\sigma}^2)} G\big(s, \mathbf{X}_s, Y_s^\alpha, Z_s^\alpha, U_s^\alpha, \alpha_s, \eta_s\big)\, ds - \int_t^T Z_s^\alpha\, dX_s^c \\ - \int_t^T \int_E U_s^\alpha(e)\cdot \tilde{\mu}_P(ds,de) - \int_t^T dK_s^\alpha.$$

Note that

$$Y_0^\alpha = \operatorname*{ess\,inf}_{\mathbb{P}'\in\mathcal{P}[\mathbb{P},\mathbb{F}^+,0]}{}^{\mathbb{P}}\, \mathcal{Y}_0^{\mathbb{P}',\alpha}, \quad \mathbb{P}\text{-a.s. for every } \mathbb{P}\in\mathcal{P}. \tag{3.6}$$

where for any $\mathbb{P}\in\mathcal{P}$ the tuple $\big(\mathcal{Y}_t^{\mathbb{P},\alpha},\ \mathcal{Z}_t^{\mathbb{P},\alpha},\ \mathcal{U}_t^{\mathbb{P},\alpha}\big)$ is the solution of the (well-posed) linear BSDEJ,

$$\mathcal{Y}_t^{\mathbb{P},\alpha} = U_A(\xi) + F^A(\mathbf{X}_T) + \int_t^T \inf_{\eta\in\mathcal{H}(x,\tilde{\sigma}^2)} G\big(s, \mathbf{X}_s, \mathcal{Y}_s^{\mathbb{P},\alpha}, \mathcal{Z}_s^{\mathbb{P},\alpha}, \mathcal{U}_s^{\mathbb{P},\alpha}, \alpha_s, \eta_s\big)\, ds \\ - \int_0^T \mathcal{Z}_s^{\mathbb{P},\alpha,\eta}\, d\mathbf{X}_s^c - \int_0^T\int_E \mathcal{U}_s^{\mathbb{P},\alpha,\eta}(e)\cdot\tilde{\mu}_P^0(ds,de), \quad \mathbb{P}\text{-a.s.} \tag{3.7}$$

**Step 2: Comparison.** From comparison theorem for the BSDEJ (3.5) and (3.7), we deduce that $\mathcal{Y}_0^{\mathbb{P},\alpha} \le \mathcal{Y}_0^{\mathbb{P},\alpha,\eta}$, for any $\mathbb{P}\in\mathcal{P}$ and the equality hold for $\eta$ optimizing the infimum. Therefore, from the representation (3.4) and (3.6) we deduce that

$$Y_0 = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} Y_0^\alpha = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} \operatorname*{ess\,inf}_{\eta\in\mathcal{H}(\hat{\sigma}^2)} Y_0^{\alpha,\eta}, \quad \mathbb{P}\text{-a.s. for every } \mathbb{P}\in\mathcal{P}. \tag{3.8}$$

**Step 3: linearization and value function.** The generator $G$ is linear in $y, z, u$. By using standard linearization tools for BSDEJ, see for example Quenez and Sulem (2013) we get

$$\mathcal{Y}_0^{\mathbb{P},\alpha,\eta} = \mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{0,T}\Big(U^A(\xi) + F^A(\mathbf{X}_T)\Big) - \int_0^T \mathcal{K}_{0,s} C^A(s,\mathbf{X}_s,\alpha_s)ds\right], \quad \mathbb{P}\text{-a.s.},\ \mathbb{P}\in\mathcal{P}_0.$$

**Step 4: characterization of the value function.** From the previous steps, it follows that $\mathbb{P}^{\alpha,\eta}\in\mathcal{P}^\alpha$ and $\mathbb{P}$-a.s. for every $\mathbb{P}\in\mathcal{P}$:

$$Y_0 = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}}{}^{\mathbb{P}} \operatorname*{ess\,inf}_{\eta\in\mathcal{H}(\hat{\sigma}^2)}{}^{\mathbb{P}} \operatorname*{ess\,inf}_{\mathbb{P}'\in\mathcal{P}[\mathbb{P},\mathbb{F}^+,0]}{}^{\mathbb{P}}\, \mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{0,T}\Big(U^A(\xi)+F^A(\mathbf{X}_T)\Big) - \int_0^T \mathcal{K}_{0,s}C^A(s,\mathbf{X}_s,\alpha_s)ds\right] \\ = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}}{}^{\mathbb{P}} \operatorname*{ess\,inf}_{(\mathbb{P}',\eta)\in\mathcal{H}^\alpha[\mathbb{P},\mathbb{F}^+,0]}{}^{\mathbb{P}}\mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{0,T}\Big(U^A(\xi)+F^A(\mathbf{X}_T)\Big) - \int_0^T \mathcal{K}_{0,s}C^A(s,\mathbf{X}_s,\alpha_s)ds\right].$$

The characterization (3.3) then follows by similar arguments to those used in the proofs of Lemma 3.5 and Theorem 5.2 of Possamaï et al. (2018).

**Step 5: optimizers.** We now turn to the second part of the theorem, where the characterization of an optimal triplet $(\alpha,\, \eta, \mathbb{P})$ for the optimization problem (3.3) is shown. From the previous steps], it is clear that a control $\big(\hat{\alpha}, \eta^\star, \mathbb{P}^\star\big)$ is optimal if and only if it attains all the essential suprema and infima above. In particular, the infimum in (3.4) is attained under conditions $(ii)$, and equality (3.8) holds if $\big(\hat{\alpha}, \eta^*\big)$ satisfy $(i)$. $\square$

# 4 Optimal contract, Perron's method and viscosity solution

Regarding Theorem 3.3, by setting $\mathcal{Z} := \mathbb{H}^p_0(\mathbb{F}^{\mathcal{P}}, \mathcal{P}) \times \mathbb{J}^p_0(\mathbb{F}^{\mathcal{P}}, \mathcal{P}) \times \mathbb{K}^p_0(\mathbb{F}^{\mathcal{P}}, \mathcal{P})$, the bilevel adversarial agency optimization becomes

$$\textbf{(2Mm-}\sigma\textbf{)} \quad V_0^P := \sup_{(Y_0,Z,U,K)\in\mathbb{R}\times\mathcal{Z}} \inf_{(\mathbb{P},\eta)\in\mathcal{P}^{\hat{\alpha}}} \mathbb{E}^{\mathbb{P}}\Big[F^P(\mathbf{X}_T) - U_A^{-1}\big(Y_T^{Y_0,Z,U,K} - F^A(\mathbf{X}_T)\big) - \int_0^T C^P(s,\mathbf{X}_s,\eta_s)ds\Big]$$

subject to

$$(R): \quad \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}[Y_0] \geq R_0,$$

where $\hat{\alpha}$ is given by Theorem 3.3. For the sake of simplicity, we are assuming that $\hat{\alpha}$ is unique, which is satisfies in linear-quadratic models for $\mathbf{b}$ and $C^A$.

Note that we are facing with one fundamental difficulty. Under the sup-inf framework, the standard DPP fails to hold. As noted in Bayraktar and Yao (2013), without compactness of the optimization domain, we can only establish a weak DPP, which does not suffice for obtaining a well-posed viscosity solution.

To address this, we employ Perron's method. The main novelty compared to the earlier work in Hernández-Santibánez and Mastrolia (2019) lies in the incorporation of the jump term.

Note that this problem can be rewritten as

$$\textbf{(2Mm-}\sigma\textbf{)} \quad V_0^P(\mathbf{x}) := \sup_{Y_0,\, \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}[Y_0]\geq R_0} V_0^P(Y_0)$$

where

$$V_0^P(Y_0) = \sup_{(Z,U,K)\in\mathcal{Z}} \inf_{(\mathbb{P},\eta)\in\mathcal{P}^{\hat{\alpha}}} \mathbb{E}^{\mathbb{P}}\Big[F^P(\mathbf{X}_T) - U_A^{-1}\big(Y_T^{Y_0,Z,U,K} - F^A(\mathbf{X}_T)\big) - \int_0^T C^P(s,\mathbf{X}_s,\eta_s)ds\Big].$$

with[1]

$$\begin{cases} d\mathbf{X}_t = \mathbf{b}(t,\mathbf{X}_t;\hat{\alpha}_t,\eta_t)dt + \boldsymbol{\sigma}(t,\mathbf{X}_t,\eta_t)d\mathbf{W}_t^\eta + \int_E L_t(\mathbf{X}_{t-},e)\tilde{\mu}_P(de,dt),\ t\in[s,T] \\ dY_t^{Y_0,Z,U,K} = [Z_t\cdot\mathbf{b}(t,\mathbf{X}_t;\hat{\alpha}_t,\eta_t) - G^*(t,\mathbf{X}_t,Y_t,Z_t,U_t,\hat{\sigma}_t^2)]dt + Z_t\cdot d\mathbf{W}_t^\eta + dK_t + \int_E U_t(e)\tilde{\mu}_P(de,dt), \\ \mathbf{X}_0 = \mathbf{x}\in\mathbb{R}^3, \\ Y_0^{Y_0,Z,U,K} = Y_0,\ \mathbb{P}-a.s.,\ \forall\mathbb{P}\in\mathcal{P}_0. \end{cases} \tag{4.1}$$

Recalling similar argument that Hernández-Santibánez and Mastrolia (2019) and in order to derive the corresponding HJB-Isaacs equation, we first note that $K$ can be regularized as follows

**Lemma 4.1.** *Without loss of generality, see (Cvitanić et al., 2018, Remark 5.1), there exists a predictable process $\Gamma$ such that*

$$K_s = \int_t^s \Big(G^\star(r,\mathbf{X}_r,Y_r,Z_r,U_r,\hat{\sigma}_r^2) + \frac{1}{2}Tr(\hat{\sigma}_r^2\Gamma_r) - \widehat{G}(r,X_r,Y_r,Z_r,U_r,\Gamma_r)\Big)dr$$

*and the solution for the 2BSDEJ with this pattern of $K$ still admits the optimal value.*

## 4.1 Integro partial HJB-Isaacs equation

We start to introduce the dynamic version of this optimization at time $t$. Let $(\mathbf{x},y)\in\mathbb{R}^3\times\mathbb{R}$, we define the dynamic version of the the value function of the holding company by

$$V_t^P(\mathbf{x},y) := \operatorname*{ess\,sup}_{(Z,U,K)\in\mathcal{Z}} \operatorname*{ess\,inf}_{(\mathbb{P},\eta)\in\mathcal{P}^{\hat{\alpha}}} \mathbb{E}^{\mathbb{P}}_{t,\mathbf{x},y}\Big[F^P(\mathbf{X}_T) - U_A^{-1}\big(Y_T^{Y_0,Z,U,K} - F^A(\mathbf{X}_T)\big) - \int_t^T C^P(s,\mathbf{X}_s,\eta_s)ds\Big].$$

[1] To alleviate the notations, we omit the super indexes in the definition of $Y$ in the next sections.

We define

$$B(t, \mathbf{x}, y; z, \gamma, \mathfrak{a}, \mathfrak{h}) := \begin{pmatrix} \mathbf{b}(t, \mathbf{x}; \mathfrak{a}, \mathfrak{h}) \\ z \cdot \mathbf{b}(t, \mathbf{x}; \mathfrak{a}, \mathfrak{h}) + \frac{1}{2}\mathrm{Tr}(\boldsymbol{\sigma}\boldsymbol{\sigma}^\top(t, \mathbf{x}, \mathfrak{h})\gamma) - \widehat{G}(t, \mathbf{x}, y, z, u, \gamma) \end{pmatrix},$$

$$\Sigma(t, \mathbf{x}, z, \mathfrak{h}) = \begin{pmatrix} \boldsymbol{\sigma}(t, \mathbf{x}, \eta) \\ z^1, z^2, z^3 \end{pmatrix},$$

and the Hamiltonian operator $\mathcal{Q}^\star$ for any $v : [0, T] \times \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\mathcal{Q}^\star[v](t, \mathbf{x}, y) := \sup_{(z,u,\gamma)\in\mathbb{R}^3\times\mathcal{L}^{2,m}_\nu} \inf_{\eta\in H} \mathcal{Q}^{z,u,\gamma,\eta}[v](t, \mathbf{x}, y)$$

with

$$\begin{aligned}\mathcal{Q}^{z,u,\gamma,\eta}[v](t, \mathbf{x}, y) : &= B(t, \mathbf{x}, y; z, \gamma, \hat{\alpha}(\mathbf{x}, z, u), \eta) \cdot \nabla v(t, \mathbf{x}, y) \\ &+ \big(v(t, \mathbf{x} + \int_E L_t(\mathbf{x}, e)\nu_t(de), y + \int_E u_t(e)\nu_t(de)) - v(t, \mathbf{x}, y)\big)\lambda(\eta, \mathbf{x}^3) \\ &+ \frac{1}{2}\mathrm{Tr}(\Sigma(t, \mathbf{x}, z, \eta)\Sigma(t, \mathbf{x}, z, \eta)^\top \Delta v(t, \mathbf{x}, y)) - C^P(t, \mathbf{x}, \eta)\end{aligned}$$

Therefore, the corresponding HJB-Isaacs equation is

$$(HJBI) \begin{cases} 0 = \partial_t v(t, \mathbf{x}, y) + \mathcal{Q}^\star[v](t, \mathbf{x}, y),\ t < T, \\ v(T, \mathbf{x}, y) = F^P(\mathbf{x}) - U_A^{-1}(y - F^A(\mathbf{x})),\ \mathbf{x} \in \mathbb{R}^3, y \in \mathbb{R}. \end{cases}$$

## 4.2 Restriction to piece-wise constant controls

To apply Perron's method, we restrict the study to elementary piece-wise constant controls.

**Definition 4.1** (Elementary controls starting at a stopping time.)**.** *Let $t \in [0, T]$ and $\tau$ be a stopping time $\mathcal{G}_t$-adapted for any $s \in [t, T]$. We say that an $\mathbb{R}^3 \times \mathbb{R}_+ \times \mathcal{L}^{p,m}_\nu$-valued process $(Z, K, U)$ (resp. $\eta \in \mathfrak{H}$) is an elementary control starting at $\tau$ for the Principal (resp. the Attacker) if there exist:*

- *a finite sequence $(\tau_i)_{0\le i\le n}$ of $\mathcal{F}_t$-adapted stopping times such that*

$$\tau = \tau_0 \le \cdots \le \tau_n = T,$$

- *a sequence $(z_i, k_i, u_i)_{1\leq i\leq n}$ of $\mathbb{R}^3\times\mathbb{R}_+\times\mathcal{L}^{p,m}_\nu$-valued random variables (functions) such that $z_i, k_i, u_i$ are $\mathcal{F}_t$-measurable with respect to $\tau_{i-1}$, and*

$$Z_t = \sum_{i=1}^n z_i \mathbf{1}_{\tau_{i-1}<t\leq\tau_i}, \quad K_t = \sum_{i=1}^n k_i \mathbf{1}_{\tau_{i-1}<t\leq\tau_i}, \quad U_t = \sum_{i=1}^n l_i \mathbf{1}_{\tau_{i-1}<t\leq\tau_i}$$

- *resp. a sequence $(h_i)_{1\leq i\leq n}$ of $H$-valued random variables such that $h_i$ is $\mathcal{F}_t$-measurable with respect to $\tau_{i-1}$, and*

$$\eta_t = \sum_{i=1}^n h_i \mathbf{1}_{\tau_{i-1}<t\leq\tau_i}.$$

*We denote by $\mathfrak{K}(t,\tau)$ (resp. $\mathfrak{H}(t,\tau)$) the set of elementary controls of the Principal (resp. the Hacker). If $\tau = t = 0$, we just write $\mathfrak{K}$ (resp. $\mathfrak{H}$). For any $(\mathbb{P},\eta)\in\mathcal{P}^\alpha$ we denote by $\mathfrak{P}^\alpha$ its restriction to $\eta\in\mathfrak{H}$.*

Then we have the optimization problem:

$$V_0^P = \sup_{Y_0,\,\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}[Y_0]\geq R_0} V_0^P(Y_0)$$

where

$$V_0^P(Y_0) := \sup_{(Z,K,U)\in\mathfrak{K}} \inf_{(\mathbb{P},\eta)\in\mathfrak{P}^{\hat{\alpha}}} \mathbb{E}\left[F^P(\mathbf{X}_T) - U_A^{-1}(Y_T^{Y_0,Z,K,U} - F^A(\mathbf{X}_T))\right]$$

**Assumption 4.1.** *For every $\phi : [0,T]\times\mathbb{R}^3\times\mathbb{R}\longrightarrow\mathbb{R}$ continuously differentiable in time and twice continuously differentiable in space and for any $(t,\mathbf{x},y)\in[0,T]\times\mathbb{R}^3\times\mathbb{R}$, there exists a continuous radius $R := R(t,\mathbf{x},y)$ such that*

$$\mathcal{Q}^\star[\phi](t,\mathbf{x},y) = \sup_{|z|\leq R}\sup_{\|u(\cdot)\|\leq R}\sup_{|\gamma|\leq R}\inf_{\eta\in H}\mathcal{Q}^{z,u,\gamma,\eta}[\phi](t,\mathbf{x},y).$$

## 4.3 Perron's method to characterize the value function as a weak solution to an HJBI-PDE

**Definition 4.2** (Stopping Rule)**.** *For $s\in[t,T]$, we define the filtration $\mathcal{B}^t_s = \sigma((\mathbf{X}_u, Y_u), t\leq u\leq s)$, $t\leq s\leq T$. We say that $\tau\in C([t,T],\mathbb{R}^4)$ is a stopping rule starting at $t$ if it is a stopping time with respect to $\mathcal{B}^t_s$.*

**Definition 4.3** (Stochastic semisolutions of (HJBI)). *Let* $v\colon [0,T]\times\mathbb{R}^3\times\mathbb{R} \longrightarrow \mathbb{R}$

- ***Sub-solution.*** *$v$ is called a* stochastic sub-solution *of the HJBI equation* $(HJBI)$ *if*

  *(i-)* *$v$ is continuous and*

  $$v(T,\mathbf{x},y) \;\leq\; F^P(\mathbf{x}) - U_A^{-1}(y - F^A(\mathbf{x})) \quad \textit{for any } (\mathbf{x},y)\in\mathbb{R}^3\times\mathbb{R},$$

  *(ii-)* *for any $t\in[0,T]$ and for any stopping rule $\tau\in\mathcal{B}^t$, there exists an* elementary control $(\widetilde{Z},\widetilde{K},\widetilde{U})\in\mathfrak{K}(t,\tau)$ *such that for* any $(Z,K,U)\in\mathfrak{K}(t,t)$, *for any* $(\mathbb{P},\eta)\in\mathcal{P}^{\hat{\alpha}}$ *and every stopping rule $\rho\in\mathcal{B}^t$ with $\tau\leq\rho\leq T$ we have*

  $$v\big(\tau',\mathbf{X}^{(\tau)}_{\tau'},Y^{(\tau)}_{\tau'}\big) \;\leq\; \mathbb{E}^{\mathbb{P}}\Big[v\big(\rho',\mathbf{X}^{(\tau)}_{\rho'},Y^{(\tau)}_{\rho'}\big)\,\Big|\,\mathcal{F}^t_{\tau'}\Big] \quad \mathbb{P}\text{-}a.s.,$$

  *where, for any* $(\mathbf{x},y,\omega)\in\mathbb{R}^4\times\Omega$,

  $$\mathbf{X}^{(\tau)} \;:=\; \mathbf{X}^{\,t,\mathbf{x},\,(Z,K,U)\otimes_\tau(\widetilde{Z},\widetilde{K},\widetilde{U}),\,\eta}, \quad Y^{(\tau)} \;:=\; Y^{\,t,y,\,(Z,K,U)\otimes_\tau(\widetilde{Z},\widetilde{K},\widetilde{U}),\,\eta},$$

  *where* $\mathbf{X}^{\,t,\mathbf{x},\,(Z,K,U)\otimes_\tau(\widetilde{Z},\widetilde{K},\widetilde{U}),\,\eta}, Y^{\,t,y,\,(Z,K,U)\otimes_\tau(\widetilde{Z},\widetilde{K},\widetilde{U}),\,\eta}$ *denotes the solution to the controlled system (4.1), with concatenated elementary strategies control $(\widetilde{Z},\widetilde{K},\widetilde{U})$ starting with $(Z,K,U)$ at time $t$, see (Sîrbu, 2014, Definition 3.1)*

  $$\tau'(\omega) \;:=\; \tau\big(\mathbf{X}^{\,t,\mathbf{x},\,(Z,K,U)\otimes_\tau(\widetilde{Z},\widetilde{K},\widetilde{U}),\,\eta}(\omega),\; Y^{\,t,y,\,(Z,K,U)\otimes_\tau(\widetilde{Z},\widetilde{K},\widetilde{U}),\,\eta}(\omega)\big),$$

  $$\rho'(\omega) \;:=\; \rho\big(\mathbf{X}^{\,t,\mathbf{x},\,(Z,K,U)\otimes_\tau(\widetilde{Z},\widetilde{K},\widetilde{U}),\,\eta}(\omega),\; Y^{\,t,y,\,(Z,K,U)\otimes_\tau(\widetilde{Z},\widetilde{K},\widetilde{U}),\,\eta}(\omega)\big).$$

  *We denote by $\mathcal{V}^-$ the set of all such* stochastic sub-solutions *to* $(HJBI)$.

- ***Super-solution.*** *$v$ is a* stochastic super-solution *of the HJBI equation* $(HJBI)$ *if*

  *(i+)* *$v$ is continuous and*

  $$v(T,\mathbf{x},y) \;\geq\; F^P(\mathbf{x}) - U_A^{-1}(y - F^A(\mathbf{x})) \quad \textit{for any } (\mathbf{x},y)\in\mathbb{R}^3\times\mathbb{R},$$

  *(ii+)* *for any $t\in[0,T]$, for any stopping rule $\tau\in\mathcal{B}^t$ and for any $(Z,K,U)\in\mathfrak{K}(t,\tau)$, there exists an* elementary control $(\widehat{\mathbb{P}},\widehat{\eta})\in\mathfrak{P}^{\hat{\alpha}}$ *such that for every $\eta\in\mathfrak{H}(t,t)$ satisfying $(\widehat{\mathbb{P}},\eta)\in\mathfrak{P}^{\hat{\alpha}}$ and for every stopping rule $\rho\in\mathcal{B}^t$ with $\tau\leq\rho\leq T$, we have*

  $$v\big(\tau',\mathbf{X}^{(\tau)}_{\tau'},Y^{(\tau)}_{\tau'}\big) \;\geq\; \mathbb{E}^{\widehat{\mathbb{P}}}\Big[v\big(\rho',\mathbf{X}^{(\tau)}_{\rho'},Y^{(\tau)}_{\rho'}\big)\,\Big|\,\mathcal{F}^t_{\tau'}\Big] \quad \widehat{\mathbb{P}}\text{-}a.s.$$

*We denote by $\mathcal{V}^+$ the set of all such* stochastic super-solutions *to* $(HJBI)$.

**Assumption 4.2.** *The sets $\mathcal{V}^+$ and $\mathcal{V}^-$ are non-empty.*

As explained in Bayraktar and Sîrbu (2014); Bayraktar and Sirbu (2012), the set $\mathcal{V}^+$ is trivially non-empty if $U_P$ is bounded above, whereas $\mathcal{V}^-$ is non-empty if $U_P$ is bounded below. We now follow Perron's method as in Hernández-Santibánez and Mastrolia (2019). Define

$$v^- := \sup_{v \in \mathcal{V}^-} v, \quad v^+ := \inf_{v \in \mathcal{V}^+} v.$$

**Theorem 4.2.** *The function $v^-$ is a lower semicontinuous viscosity super-solution of the HJBI equation $(HJBI)$, and $v^+$ is an upper semicontinuous viscosity sub-solution of $(HJBI)$.*

**Remark 6.** *The proof of Theorem 4.2 is almost the same as the proof of Theorem 3.5 in Sîrbu (2014) and Theorem 4.1 in Hernández-Santibánez and Mastrolia (2019), with the only differences being that we introduce an additional elementary control tuple, i.e. $l_t$ related to the Levy process. Since these adjustments do not make a significant difference in the proof, we omit the full details here and instead refer the reader to the underlying argument.*

As a consequence of this theorem, we have the following characterization of the value function of the holding company.

**Corollary 4.2.1.** *If there is a* comparison result *for $(HJBI)$, that is, for any lower semi-continuous viscosity subsolution $\underline{v}$ and any upper semi-continuous viscosity supersolution $\overline{v}$, one has*

$$\sup_{[0,T]\times\mathbb{R}^3\times\mathbb{R}} (\underline{v} - \overline{v}) \;=\; \sup_{\mathbb{R}^3\times\mathbb{R}} \Big(\underline{v}(T,\cdot,\cdot) \;-\; \overline{v}(T,\cdot,\cdot)\Big),$$

*then $v^- = v^+$ is the unique viscosity solution of $(HJBI)$. Consequently, $V_t^P(\mathbf{x}, y)$ is the unique viscosity solution to $(HJBI)$.*

*Proof.* The equality $v^- = v^+$ is a direct consequence of Theorem 4.2 together with the comparison result assumption. Definition 4.3 it follows that for any $t \in [0,T]$,

$$v^-(t,\mathbf{x},y) \le V_t^P(\varphi) \le v^+(t,\mathbf{x},y).$$

As a consequence, $V_t^P(\mathbf{x}, y) = v^-(t, \mathbf{x}, y) = v^+(t, \mathbf{x}, y)$ and therefore, it is the unique viscosity solution to $(HJBI)$. $\square$

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# A Spaces

Let $\mathbb{X} := (\mathcal{X}_s)_{t\leq s\leq T}$ denote an arbitrary filtration on $(\Omega, \mathcal{F}_T)$, and let $\mathbb{P}$ be an arbitrary element in $\mathcal{P}(t,\omega)$. We follow the notations of spaces in Hernández-Santibánez and Mastrolia (2019); Possamaï et al. (2018); Denis et al. (2024).

- **The spaces $\mathbb{L}^{p,\kappa}_{t,x}$.** For each $p \geq \kappa \geq 1$, we define $\mathbb{L}^p_{t,\omega}(\mathbb{X})$ (resp. $\mathbb{L}^p_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathcal{X}_T-$measurable random variables $\xi$ such that

$$\|\xi\|_{\mathbb{L}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left(\mathbb{E}^{\mathbb{P}}[|\xi|^p]\right)^{1/p} < +\infty, \text{ resp. } \|\xi\|_{\mathbb{L}^p_{t,\omega}(\mathbb{P})} := \left(\mathbb{E}^{\mathbb{P}}[|\xi|^p]\right)^{1/p} < +\infty.$$

  We set

$$\mathbb{L}^{p,\kappa}_{t,\omega}(\mathbb{X}) := \left\{\xi \in \mathbb{L}^p_{t,\omega}(\mathbb{X}) : \|\xi\|_{\mathbb{L}^{p,\kappa}_{t,\omega}} < \infty\right\},$$

  where the norm is given by

$$\|\xi\|_{\mathbb{L}^{p,\kappa}_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left(\mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{t\leq s\leq T}\left(\mathbb{E}^{\mathbb{P}}_{t,\omega,\mathcal{X}_s^+}[|\xi|^\kappa]\right)^{\frac{p}{\kappa}}\right]\right)^{\frac{1}{p}}.$$

- **The spaces $\mathbb{H}^p_{t,x}(\mathbb{X},\mathbb{P})$.** We say $Z$ is in $\mathbb{H}^p_{t,x}(X,\mathbb{P})$ if $Z$ is an $X$-predictable, $\mathbb{R}^d-$valued process satisfying

$$\|Z\|^p_{\mathbb{H}^p_{t,x}(\mathbb{X},\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \|\sigma_s^{\frac{1}{2}} Z_s\|^2\, ds\right)^{\frac{p}{2}}\right] < +\infty.$$

We then define

$$\mathbb{H}^p_{t,x}(X,\mathcal{P}) := \Big\{ Z : \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \|Z\|_{\mathbb{H}^p_{t,x}(\mathbb{X},\mathbb{P})} < +\infty \Big\}.$$

- **The spaces $\mathbb{S}^p_{t,x}(\mathbb{X},\mathbb{P})$.** We say $Y$ is in $\mathbb{S}^p_{t,x}(\mathbb{X},\mathbb{P})$ if $Y$ is an $\mathbb{X}$-progressively measurable, real-valued process satisfying

$$\|Y\|^p_{\mathbb{S}^p_{t,x}(\mathbb{X},\mathbb{P})} := \mathbb{E}^P\Big[ \sup_{s\in[t,T]} |Y_s|^p \Big] < +\infty.$$

We then define

$$\mathbb{S}^p_{t,x}(\mathbb{X},\mathcal{P}) := \Big\{ Y : \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \|Y\|_{\mathbb{S}^p_{t,x}(\mathbb{X},\mathbb{P})} < +\infty \Big\}.$$

- **The space $\mathbb{J}^p_{t,\omega}(\mathbb{X})$.** $\mathbb{J}^p_{t,\omega}(\mathbb{X})$ (resp. $\mathbb{J}^p_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathbb{X}$-predictable functions $U$ such that

$$\|U\|_{\mathbb{J}^p_{t,\omega}(\mathbb{X})} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left( \mathbb{E}^{\mathbb{P}}\left[ \left( \int_t^T \int_E \|U_s(e)\|^2 \, \nu_s(de)\, ds \right)^{p/2} \right] \right)^{1/p} < +\infty,$$

resp.

$$\|U\|_{\mathbb{J}^p_{t,\omega}(\mathbb{X},\mathbb{P})} := \left( \mathbb{E}^{\mathbb{P}}\left[ \left( \int_t^T \int_E \|U_s(e)\|^2 \, \nu_s(de)\, ds \right)^{p/2} \right] \right)^{1/p} < +\infty.$$

- **The spaces $\mathbb{K}^p_{t,x}(\mathbb{X},\mathbb{P})$.** We say $K$ is in $\mathbb{K}^p_{t,x}(\mathbb{X},\mathbb{P})$ if $K$ is an $\mathbb{X}$-optional, real-valued process with $\mathbb{P}-$a.s. càdlàg, non-decreasing paths on $[t,T]$, $K_t = 0$ $\mathbb{P}-$a.s., and

$$\|K\|^p_{\mathbb{K}^p_{t,x}(\mathbb{X},\mathbb{P})} := \mathbb{E}^P\big[ |K_T|^p \big] < +\infty.$$

We denote by $\mathbb{K}^p_{t,x}\big(\mathbb{X},\mathcal{P}\big)$ the set of all families $(K^P)_{P\in\mathcal{P}(t,x)}$ such that $K^P \in \mathbb{K}^p_{t,x}(\mathbb{X},\mathbb{P})$ for every $\mathbb{P}\in\mathcal{P}(t,x)$ and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,x)} \|K^{\mathbb{P}}\|_{\mathbb{K}^p_{t,x}(\mathbb{X},\mathbb{P})} < +\infty.$$

- **The spaces $\mathcal{L}^{p,m}_\nu$.**
We define $\mathcal{L}^{p,m}_\nu$ as the set of Borel measurable functions $\ell : \mathbb{R}^* \to \mathbb{R}^m$ satisfying

$$\|\ell\|_{p,\nu} := \int_{\mathbb{R}^*} \|\ell(u)\|^p \, \nu(du) < +\infty.$$