# Improving Statistical Privacy by Subsampling<sup>\*</sup>

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**Abstract.** Differential privacy (DP) considers a scenario, where an adversary has almost complete information about the entries of a database This worst-case assumption is likely to overestimate the privacy thread for an individual in real life. Statistical privacy (SP) denotes a setting where only the distribution of the database entries is known to an adversary, but not their exact values. In this case one has to analyze the interaction between noiseless privacy based on the entropy of distributions and privacy mechanisms that distort the answers of queries, which can be quite complex.

A privacy mechanism often used is to take samples of the data for answering a query. This paper proves precise bounds how much different methods of sampling increase privacy in the statistical setting with respect to database size and sampling rate. They allow us to deduce when and how much sampling provides an improvement and how far this depends on the privacy parameter  $\varepsilon$ . To perform these investigations we develop a framework to model sampling techniques.

For the DP setting tradeoff functions have been proposed as a finer measure for privacy compared to  $(\varepsilon, \delta)$ -pairs. We apply these tools to statistical privacy with subsampling to get a comparable characterization

Keywords: privacy  $\cdot$  sampling  $\cdot$  tradeoff function

# 1 Introduction

Many machine learning algorithms, such as stochastic optimization methods and Bayesian inference algorithms include sampling operations. Due to the increasing demand for such learning algorithms, especially for sensitive data, there have been recent efforts to investigate the influence of sampling methods with respect to privacy [1,17,2,12,16]. However, the standard model *differential privacy* makes extremely strong assumptions about the power of an adversary that from answers of queries to a database tries to get information of single entries. To guarantee privacy in this case a strong distortion of the answers is necessary – thus the quality deteriorates significantly [13,6]. For many real life applications one should consider less powerful adversaries in the hope to improve the utility

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of the data. The prior information, also called background knowledge is limited. Such settings has been propsed and named *noiseless privacy*, *(inference based) distributional differential privacy* and *passive partial knowledge differential privacy* [4,3,7]. Such a model requires a more complex mathematical analysis than a worst case scenario.

This paper consider a situation, where only the distribution of the database entries is publicly known, but not their exact values nor any further information, which we call *Statistical privacy* [5]. To consider different types of sampling methods are considered for increasing statistical privacy we develop a general framework using *sampling templates*. Combining this with Markov process theory we analyze how the distributions generated effect privacy and define *statistical privacy curves*. Then for the analysis we use tools developed by Balle et. al. in [1] and Markov process theory to bound the  $\alpha$ -divergence. This establishes precise bounds for the amplification of privacy by sampling techniques depending on database size and sampling rate. In addition to the analytical estimation we plots are provided for different parameter settings that illustrate the improvement, in particular with respect to the privacy parameter  $\varepsilon$ .

For the DP setting tradeoff functions and the notion of f-differential privacy have been proposed by Dong et. al in [9] as a finer measure for privacy. We apply these tools to statistical privacy with subsampling to get a comparable characterization.

Our main results are the following:

- a general framework for sampling techniques
- a simple analytical formular for the amplification by sampling without replacement
- an analysis for Poission sampling in the statistical setting that handles different sample sizes
- managing dependencies appearing in case of sampling with replacement

This paper is organized as follows. In the next section formal definitions. In section 3 we develop the framework for sampling mechanisms and introduce sampled privacy curves. The bounds for the different sampling techniques are proven in section 4. In the next section tradeoff functions are transferred to statistical privacy with subsampling. The paper concludes with....

An appendix provides further details and an introduction to the basics of stochastic process theory used in our proofs.

# 2 Preliminaries

In this paper we will use the following notions.

**Definition 1.** A database D of size n is a sequence of n independent entries  $I_1, \ldots, I_n$  where  $I_j$  is taken from a set W of possible values for entries.  $I_j$  is distributed according to a distribution  $\overline{\mu_j}$  on W. These marginal distributions may be different, but their support has to be identical – thus we assume that it is equal to W. They imply a distribution, resp. density function  $\mu$  on  $W^n$ . This

is modeled by a random variable  $\mathcal{D}$  on  $W^n$  that is distributed according to  $\mu$  $(D \sim \mu)$ .

By  $\mu_{j,w}$  and  $\mathcal{D}_{j,w}$  we denote the conditional distribution, resp. random variable where  $I_j$  is fixed to a constant value  $w \in W$ .

Let  $\mathcal{F}$  be a set of queries that may be asked for a given database. Formally, this is described by measurable functions  $\mathbf{F} : \mathbf{W}^n \mapsto \mathbf{A}$ , where A denotes an appropriate set of possible answers. We assume that W and A are totally ordered and call a query monotone if  $D \leq \hat{D}$  implies  $F(D) \leq F(\hat{D})$  where the partial order on  $W^n$  is defined by  $I_j \leq \hat{I}_j$  for all j.

A privacy technique  $\mathcal{M}$  (also called mechanism) is a function that maps database queries to random variables on A, that means for a given database D and query F,  $\mathcal{M}(F, D)$  is a random variable on A that disturbs the correct answer F(D). If the database itself is a random variable  $\mathcal{D}$  then  $\mathcal{M}(F, \mathcal{D})$  contains two types of randomness, the internal uncertainty of  $\mathcal{D}$  and the external noise imposed by  $\mathcal{M}$ .  $\square$ 

To measure how much privacy of entries is lost by a query F one considers a critical entry j and estimates how much information about its value  $I_j$  an adversary can deduce from the answer. The most challenging task would be to reconstruct  $I_j$  or part of it. Privacy investigations typically consider a much simpler decision problem, namely to differentiate between two neighboring databases, namely whether the entry  $I_j$  equals v or w for values in  $v, w \in W$  while keeping all other entries unchanged. Alternatively, instead of changing the j-th entry one could remove it to get a different notion of neighborhood. But this does not make much difference. When studying sampling techniques it is more convenient not to change the database size, thus we will only consider replacement of the j-th entry by a different value.

### Definition 2. Differential Privacy (DP)[10]

For a collection  $\mathcal{W}$  of databases a privacy technique M achieves  $(\varepsilon, \delta)$ -differential privacy for a query F if for all neighboring databases D, D' in  $\mathcal{W}$  and for all  $S \subseteq A$  it holds

$$\Pr[M(F, D) \in S] \leq e^{\varepsilon} \Pr[M(F, D') \in S] + \delta.$$

To protect privacy one should not allow queries that differentiate between entries. Furthermore, sampling only makes sense if the random order in which elements are drawn does not matter. If one requires that queries F are symmetric this holds. But in the distributional setting one can be more general. We consider functions of the form

$$F(x_1,\ldots,x_n) = h(g_1(x_1),\ldots,g_n(x_n))$$

where  $h: W' \to A$  for some set W' is symmetric, but the  $g_j: W \to W'$ can be arbitrary. The distribution  $\mu$  for vectors  $(x_1, \ldots, x_n) \in W^n$  induces a distribution  $\mu'$  for  $(y_1, \ldots, y_n) \in W'^n$  where  $y_i = g_i(x_i)$ . Thus, such a query Fon  $W^n$  with distribution  $\mu$  is equivalent to a symmetric query  $F' = h(y_1, \ldots, y_n)$ on  $W'^n$  with distribution  $\mu'$ .

Taking samples from a sequence may generate arbitrary orderings of elements. Reordering a sample should have no influence. Therefore, a distance measure for database distributions and samples should be invariant against permutations.

#### **Definition 3. Distributional Distance**

For distributions  $\mu = \overline{\mu}_1 \times \ldots \times \overline{\mu}_n$  and  $\nu = \overline{\nu}_1 \times \ldots \times \overline{\nu}_n$  of databases their distance is defined as

$$\Gamma(\mu,\nu) := \min_{\pi \in \Pi_n} |\{j \mid \overline{\mu}_j \neq \overline{\nu}_{\pi(j)}\}| ,$$

where  $\Pi_n$  denotes the set of all permutations on n elements.

If the sizes n, n' of the databases differ the distance  $\Gamma(\mu, \nu)$  is computed by minimizing over all injective functions  $\pi$  from the smaller index set to the larger one and adding n - n' to the value obtained.

### Definition 4.

Let  $\mu, \hat{\mu}$  be distributions and F a query. The the corresponding **Privacy Loss Random Variable** (PLRV) [8] is a real-valued function defined for  $a \in A$  as

$$\mathcal{L}_{\mu,\hat{\mu}}^{F}(a) := \ln \frac{\mu \mathcal{K}_{F}(a)}{\hat{\mu} \mathcal{K}_{F}(a)} ,$$

where  $\mathcal{K}_F$  denotes the Markov kernel corresponding to F and  $\ln \frac{0}{0} := 0$  and  $\ln \frac{>0}{0} := \infty$ . This operator maps a distribution  $\mu$  on  $W^n$  to the distribution on A that is generated by F.

Using the PLRV, a distance measure for  $\mu \mathcal{K}_F$  and  $\hat{\mu} \mathcal{K}_F$  called **privacy curve** [1] has been defined as

$$\begin{split} \Delta_{F,\mu,\hat{\mu}}(\varepsilon) &:= \int_{A} \mu \mathcal{K}_{F}(a) \cdot \max\left(0, \ 1 - \exp\left(\varepsilon - \mathcal{L}_{\mu,\hat{\mu}}^{F}(a)\right)\right) \ \mathbf{d}a \\ &= \int_{A} \max\left(0, \ \mu \mathcal{K}_{F}(a) - e^{\varepsilon} \ \hat{\mu} \mathcal{K}_{F}(a)\right) \ \mathbf{d}a \\ &= \max_{S \subseteq A} \ \int_{S} \mu \mathcal{K}_{F}(a) - e^{\varepsilon} \ \hat{\mu} \mathcal{K}_{F}(a) \ \mathbf{d}a \qquad \text{where } \varepsilon \geq 0. \end{split}$$

Instead of distributions we may also use random variables  $\mathcal{D}$  with  $\mathcal{D} \sim \mu$ and define the distance betweren these random variables with respect to F by  $\Delta_{F,\mathcal{D},\hat{\mathcal{D}}} := \Delta_{F,\mu,\hat{\mu}}$ .

This distance is also known as  $\alpha$ -divergence with  $\alpha = e^{\varepsilon}$ , see for example [1].

# Definition 5. Statistical Privacy (SP)[5]

A privacy technique M achieves  $(\varepsilon, \delta)$ -statistical privacy with respect to a distribution  $\mu$  (or a collection of distributions) and a query F if (for all  $\mu$ ) for every subset  $S \subseteq A$ , every entry j and all  $v, w \in W$  it holds for  $\mathcal{D} \sim \mu$ :

$$\Pr\left[M(F, \mathcal{D}_{j, v}) \in S\right] \leq e^{\varepsilon} \Pr\left[M(F, \mathcal{D}_{j, w}) \in S\right] + \delta .$$

The statistical privacy curve with respect to  $\mu$  and F generated by M is the function  $\Phi_{\mu,F,M}$ , where  $\Phi_{\mu,F,M}(\varepsilon)$  denote the smallest  $\delta$  such that M for F and  $\mu$  achieves  $(\varepsilon, \delta)$ -statistical privacy. In case that no operator M is used we simply write  $\Phi_{\mu,F}$  – thus privacy is only generated by the distribution  $\mu$ .

In the following we assume that W is completely ordered and consider the partial order on  $W^n$  defined by  $D \leq D'$  iff for all i holds  $D_i \leq D'_i$ . The answer set A for queries is restricted to real numbers and a query F is monotone if  $D \leq D'$  implies  $F(D) \leq F(D')$ .

# 3 Subsampling

Taking a sample from a groundset can be done in different ways – a fixed size sample drawn with or without replacement, or with varying size as in Poisson sampling. Given a function F with n arguments to estimate  $F(x_1, \ldots, x_n)$  using a sample  $y_1, \ldots, y_m$  one needs a family of functions  $\mathcal{F} = (F_m(y_1, \ldots, y_m))_{m \in \mathbb{N}}$  that approximate F. As already discussed above this only make sense for symmetric functions, in particular since a sample may occur in an arbitrary order. Thus in the following when we talk about a query F we actually mean the family  $\mathcal{F}$  and depending of the size of the sample the appropriate member. Classical examples for such families are the mean or the median of a sequence of entries. On the other hand, functions like min or max are not reasonable candidates for sampling unless special restrictions are put on the distributions. If a distribution is not smooth, but there are very few entries that determine the extrema it is unlikely that such an element is drawn and therefore one cannot hope for an acceptable approximation.

### **Definition 6. Subsampling**

Given a sequence  $X = x_1, \ldots, x_n$  of fixed length n, a sample  $Y = y_1, \ldots, y_m$  of X where  $y_j = x_{i_j}$  can be described by the sequence of indices  $i_j$  called a sampling template  $\tau$ . Let  $\mathfrak{C}^m$  denote the set of all such templates of size  $m \ge 0$  and  $\mathfrak{C}$  the union of all  $\mathfrak{C}^m$ . The subset of  $\mathfrak{C}$  where an entry j is drawn exactly k times is denoted by  $\mathfrak{C}_{j,k} := \{\tau \in \mathfrak{C} \mid \#\{i \mid \tau_i = j\} = k\}$  and  $\mathfrak{C}_{j,+} := \bigcup_{k=1}^{\infty} \mathfrak{C}_{j,k}$ .

For templates  $\tau, \hat{\tau}$  of identical size the relation  $\tau \approx_j \hat{\tau}$  is defined by the condition:  $\tau_i \neq j$  implies  $\tau_i = \hat{\tau}_i$  and  $\tau_i = j$  implies  $\hat{\tau}_i \neq j$ .

For fixed length n, each specific sampling technique corresponds to a random variable  $\mathcal{T} \sim \mu_{\mathcal{T}}$  on  $\mathfrak{C}$ . Conditioning  $\mathcal{T}$  on the number of times a specific entry is drawn will be denoted by  $\mathcal{T}_{j,k} \sim \mu_{\mathcal{T}} \mid \mathfrak{C}_{j,k}$  and conditioning  $\mathcal{T}$  on the size of subsampled database will be denoted by  $\mathcal{T}^m \sim \mu_{\mathcal{T}} \mid \mathfrak{C}^m$ . Furthermore conditioning  $\mathcal{T}$  on the event that a specific entry j is drawn at least once will be denoted by  $\mathcal{T}_{j,+} \sim \mu_{\mathcal{T}} \mid \mathfrak{C}_{j,+}$  and the negation by  $\mathcal{T}_{j,-} \sim \mu_{\mathcal{T}} \mid \mathfrak{C}_{j,0}$ . If j is clear from the context we will simply write  $\mathcal{T}_+$  and  $\mathcal{T}_-$ .

For a template  $\tau$  of length m, the operator  $\text{SAMP}_{\tau} : W^n \to W^m$  maps databases of size n to databases of size m, which can de described by a Markov kernel  $\mathcal{K}_{\tau}$ . Combining it with a sampling technique  $\mathcal{T}$  we get an operator  $\text{SAMP}_{\mathcal{T}}$ 

from  $W^n$  to  $W^{\star}$  with kernel

$$\mathcal{K}_{\mathcal{T}} = \int_{\mathfrak{C}} \mathcal{K}_{\tau} \, \mathbf{d} \mu_{\mathcal{T}}(\tau) \; .$$

For example, drawing a random subsample of size m without replacement corresponds to the uniform distribution on the set of all injective  $\tau$  of size m. For Poisson sampling templates of all sizes between 0 and n can occur. For each size m all injective templates have the same probability.

**Lemma 1.** For a database distribution  $\mu$ , a query F, a sampling technique  $\mathcal{T}$  and  $S \subseteq A$  it holds

$$\mu \, \mathcal{K}_{\mathcal{T}} \, \mathcal{K}_F(S) \; = \; \sum_{\tau} \; \mu_{\mathcal{T}}(\tau) \; \mu \, \mathcal{K}_{\tau} \; \mathcal{K}_F(S) \; .$$

Proof.

$$\begin{split} & \mu \, \mathcal{K}_{\mathcal{T}} \, \mathcal{K}_{F}(S) = \int_{W^{*}} \mathcal{K}_{F}(S, w) \, \mathbf{d}\mu \mathcal{K}_{\mathcal{T}}(w) \; = \; \int_{W^{*}} \mathcal{K}_{\mathcal{T}} \mathcal{K}_{F}(S, w) \, \mathbf{d}\mu(w) \\ & = \int_{W^{*}} \int_{\mathfrak{C}} \mathcal{K}_{\tau} \mathcal{K}_{F}(S, w) \, \mathbf{d}\mu_{\mathcal{T}}(\tau) \, \mathbf{d}\mu(w) \; = \; \int_{\mathfrak{C}} \int_{W^{*}} \mathcal{K}_{\tau} \mathcal{K}_{F}(S, w) \, \mathbf{d}\mu(w) \, \mathbf{d}\mu_{\mathcal{T}}(\tau) \\ & = \int_{\mathfrak{C}} \mu \mathcal{K}_{\tau} \mathcal{K}_{F}(S) \, \mathbf{d}\mu_{\mathcal{T}}(\tau) \; = \; \sum_{\tau} \; \mu_{\mathcal{T}}(\tau) \; \mu \; \mathcal{K}_{\tau} \; \mathcal{K}_{F}(S) \; . \end{split}$$

**Lemma 2.** For databases  $\mathcal{D}, \hat{\mathcal{D}}$ , subsampling techniques  $\text{SAMP}_{\mathcal{T}}, \text{SAMP}_{\mathcal{T}'}$  and a coupling  $\nu_{\mathcal{T},\mathcal{T}'}$  of  $\mathcal{T}, \mathcal{T}'$  it holds

$$\Delta_{F, \mathsf{SAMP}_{\mathcal{T}}(\mathcal{D}), \mathsf{SAMP}_{\mathcal{T}'}(\hat{\mathcal{D}})}(\varepsilon) \leq \sum_{\tau, \tau'} \nu_{\mathcal{T}, \mathcal{T}'}(\tau, \tau') \ \Delta_{F, \mathsf{SAMP}_{\tau}(\mathcal{D}), \mathsf{SAMP}_{\tau'}(\hat{\mathcal{D}})}(\varepsilon)$$

*Proof.* This can be calculated directly by

$$\begin{split} & \Delta_{F, \mathsf{SAMP}_{\mathcal{T}}(\mathcal{D}), \mathsf{SAMP}_{\mathcal{T}'}(\hat{\mathcal{D}})}(\varepsilon) = \int_{A} \max(\ 0, \ \mu \mathcal{K}_{\mathcal{T}} \mathcal{K}_{F}(a) - e^{\varepsilon} \ \hat{\mu} \mathcal{K}_{\mathcal{T}'} \mathcal{K}_{F}(a)) \ \mathbf{d}a \\ &= \int_{A} \max\left(\ 0, \ \sum_{\tau, \tau'} \nu_{\mathcal{T}, \mathcal{T}'}(\tau, \tau') \mu \mathcal{K}_{\tau} \mathcal{K}_{F}(a) - e^{\varepsilon} \cdot \sum_{\tau, \tau'} \nu_{\mathcal{T}, \mathcal{T}'}(\tau, \tau') \hat{\mu} \mathcal{K}_{\tau'} \mathcal{K}_{F}(a) \right) \ \mathbf{d}a \\ &\leq \sum_{\tau, \tau'} \nu_{\mathcal{T}, \mathcal{T}'}(\tau, \tau') \int_{A} \max(\ 0, \ \mu \mathcal{K}_{\tau} \mathcal{K}_{F}(a) - e^{\varepsilon} \cdot \hat{\mu} \mathcal{K}_{\tau'} \mathcal{K}_{F}(a)) \ \mathbf{d}a \\ &= \sum_{\tau, \tau'} \nu_{\mathcal{T}, \mathcal{T}'}(\tau, \tau') \Delta_{F, \mathsf{SAMP}_{\tau}(\mathcal{D}), \mathsf{SAMP}_{\tau'}(\hat{\mathcal{D}})}(\varepsilon) \end{split}$$

In the SP setting privacy is generated by the uncertainty of an adversary about the exact values of the entries. When taking a random sample of entries the sensitive entry may occur more than once in case of drawing with replacement or the size of the sample may vary in case of Poisson sampling. But smaller size samples have less entropy. Therefore we have to modify the notion of privacy curve by taking the expectation over all possible samples. This is legitimate since an adversary does not know which sample is actually drawn.

### Definition 7. Sampling Privacy Curve (SPC)

For a database  $\mathcal{D}$ , a query F and a sampling technique  $\mathcal{T}$  the sampling privacy curves are defined by

$$\begin{split} & \mathrm{SPC}_{F,\mathcal{D},\mathcal{T}}^{j}(\varepsilon) := \mathbf{E}_{\tau \sim \mathcal{T}_{j,+}} \left[ \max_{v,w \in W} \Delta_{F, \ \mathrm{SAMP}_{\tau}(\mathcal{D}_{j,v}), \ \mathrm{SAMP}_{\tau}(\mathcal{D}_{j,w})}(\varepsilon) \right] \ , \\ & \mathrm{SPC}_{F,\mathcal{D},\mathcal{T}}(\varepsilon) := \max_{j} \ \mathrm{SPC}_{F,\mathcal{D},\mathcal{T}}^{j}(\varepsilon) \ . \end{split}$$

**Lemma 3.** For  $\mathcal{D}, F, \nu_{\mathcal{T}, \hat{\mathcal{T}}}$  as above and  $v, w \in \text{supp}(\mathcal{D}_j)$  it holds

$$\Delta_{F, \text{ samp}_{\mathcal{T}}(\mathcal{D}_{j,v}), \text{ samp}_{\hat{\mathcal{T}}}(\mathcal{D}_{j,w})}(\varepsilon) \leq \mathtt{SPC}^{j}_{F,\mathcal{D},\nu_{\mathcal{T},\hat{\mathcal{T}}}}(\varepsilon) \; .$$

*Proof.* Let  $\mathcal{D} \sim \mu$ , this is a direct consequence of Lemma 2 and the SPC definition

**Lemma 4.** For a distribution  $\mu$ , a query F and  $v \in \operatorname{supp}(\overline{\mu_i})$  it holds

$$\Delta_{F,\mu_{j,v},\mu}(\varepsilon) \leq \max_{w \in \operatorname{supp}(\overline{\mu_j})} \Delta_{F,\mu_{j,v},\mu_{j,w}}(\varepsilon) \; .$$

*Proof.* For the density function  $\mu \mathcal{K}_F$  of  $F(\mathcal{D})$  the law of total probability gives

$$\mu \mathcal{K}_F(a) = \int_W \mu_{j,w} \mathcal{K}_F(a) \cdot \overline{\mu_j}(w) \, \mathrm{d}w \; .$$

Thus,

$$\begin{split} &\Delta_{F,\mu_{j,v},\mu}(\varepsilon) = \int_{A} \max\left(0, \ \mu_{j,v}\mathcal{K}_{F}(a) - e^{\varepsilon} \ \mu\mathcal{K}_{F}(a)\right) \ \mathbf{d}a \\ &= \int_{A} \max\left(0, \ \int_{W} \overline{\mu_{j}}(w) \ \mu_{j,v}\mathcal{K}_{F}(a) \ \mathbf{d}w - e^{\varepsilon} \ \int_{W} \mu_{j,w}\mathcal{K}_{F}(a) \ \cdot \ \overline{\mu_{j}}(w) \ \mathbf{d}w \ \right) \ \mathbf{d}a \\ &\leq \int_{W} \overline{\mu_{j}}(w) \ \int_{A} \max\left(0, \ \mu_{j,v}\mathcal{K}_{F}(a) - e^{\varepsilon} \ \mu_{j,w}\mathcal{K}_{F}(a)\right) \ \mathbf{d}a \ \mathbf{d}w \\ &\leq \max_{w \in \operatorname{supp}(\overline{\mu_{j}})} \ \underbrace{\left\{\int_{A} \max\left(0, \ \mu_{j,v}\mathcal{K}_{F}(a) - e^{\varepsilon} \ \mu_{j,w}\mathcal{K}_{F}(a)\right) \ \mathbf{d}a \ }_{=\Delta_{F,\mu_{j,v},\mu_{j,w}}(\varepsilon)} \ \underbrace{\int_{W} \overline{\mu_{j}}(w) \ \mathbf{d}w}_{=1}^{\bullet} . \end{split}$$

When changing a single entry in a database, drawing with replacement can generate samples that differ at more than 1 position, namely when the critical entry j is drawn several times, let us say k times. For the following analysis we need a smoothness condition for sampling.

### **Definition 8.** *F*-samplable

With respect to a query F, a database distribution  $\mu$  is called F-samplable if

for all sampling templates  $\tau, \hat{\tau} \in \mathfrak{C}^n$ ,  $v, w \in W$  and  $\varepsilon \geq 0$  a set  $S \subseteq A$  that maximizes the integral

$$\int_{S} \mu_{j,v} \mathcal{K}_{\tau} \mathcal{K}_{F}(a) - e^{\varepsilon} \mu_{j,w} \mathcal{K}_{\hat{\tau}} \mathcal{K}_{F}(a) \mathbf{d} a$$

can be chosen as a half open real interval  $(\infty, \hat{a}], [\hat{a}, \infty)$  or the empty set.

For example, for arbitrary monotone queries F, one gets a F-samplable distribution if the individual distributions of the entries are binomial, normal or Laplace distributed.

**Lemma 5.** Let F be a monotone query,  $\mathcal{D} \sim \mu$  a F-samplable distribution, and  $\tau, \hat{\tau} \in \mathfrak{C}^n$  two sampling templates with  $\tau \approx_j \hat{\tau}$ . Then for  $v \in W$  holds

$$\Delta_{F, \text{ samp}_{\tau}(\mathcal{D}_{j,v}), \text{ samp}_{\hat{\tau}}(\mathcal{D})}(\varepsilon) \leq \max_{w \in W} \Delta_{F, \text{ samp}_{\tau}(\mathcal{D}_{j,v}), \text{ samp}_{\tau}(\mathcal{D}_{j,w})}(\varepsilon) \ .$$

*Proof.* Consider the distributional distance  $k = \Gamma(\mu_{j,v} \mathcal{K}_{\tau}, \mu \mathcal{K}_{\hat{\tau}})$ , for k = 0 the *j*-th entry does not affect the privacy curve therefore the inequality holds and k = 1 is handled by the previous lemma. For k > 1 w.l.o.g. consider  $w_1, \ldots, w_{\ell} \in W^{\ell}$  such that  $F_{w_1,\ldots,w_{\ell}}(x_1,\ldots,x_m) := F(w_1,\ldots,w_{\ell},x_{\ell+1},\ldots,x_m)$ . Let  $S' \subseteq A$  be a set that maximizes

$$\Delta_{F, \text{ samp}_{\tau}(\mathcal{D}_{j,v}), \text{ samp}_{\hat{\tau}}(\mathcal{D})}(\varepsilon) = \max_{S \subseteq A} \int_{S} \mu_{j,v} \mathcal{K}_{\tau} \mathcal{K}_{F}(a) - e^{\varepsilon} \ \mu \mathcal{K}_{\hat{\tau}} \mathcal{K}_{F}(a) \ \mathbf{d}a.$$

Since  $\mu$  is *F*-samplable *S'* is equal to either  $(\infty, \hat{a}], [\hat{a}, \infty)$  or  $\emptyset$  for some  $\hat{a} \in \mathbb{R}$ . In case of  $\emptyset$  the inequality obviously holds. Consider the case  $S' = [\hat{a}, \infty)$ - the other one can be dealt with in an analogous way. To prove the inequality we have to show  $\mu \mathcal{K}_{\hat{\tau}} \mathcal{K}_F([\hat{a}, \infty)) \geq \mu_{j,w} \mathcal{K}_{\tau} \mathcal{K}_F([\hat{a}, \infty))$  Since *F* is monotone there exists a  $w \in W$  such that  $\mathbf{E}[F(\mathsf{SAMP}_{\hat{\tau}}(\mathcal{D}))] \geq \mathbf{E}[F_w(\mathsf{SAMP}_{\hat{\tau}}(\mathcal{D}))]$ . This gives

$$\int_{\hat{a}}^{\infty} \mu \mathcal{K}_{\hat{\tau}} \mathcal{K}_{F}(a) \, \mathbf{d}a = \int_{W^{m}} \mu \mathcal{K}_{\hat{\tau}}(x) \mathbf{1}_{[\hat{a},\infty)}(F(x)) \, \mathbf{d}x$$
$$\geq \min_{w \in W} \int_{W^{m}} \mu \mathcal{K}_{\hat{\tau}}(x) \mathbf{1}_{[\hat{a},\infty)}(F_{w}(x)) \, \mathbf{d}x$$

By iteration it holds

$$\mu \mathcal{K}_{\hat{\tau}} \mathcal{K}_F([\hat{a},\infty)) \ge \min_{(w_1,\dots,w_k)} \int_{W^m} \mu \mathcal{K}_{\hat{\tau}}(x) \mathbf{1}_{[\hat{a},\infty)}(F_{w_1,\dots,w_k}(x)) \, \mathbf{d}x$$

For a symmetric function F the vector  $(w_1, \ldots, w_k)$  that minimizes this integral has identical components. This can be used to derive the following bound

$$\begin{split} \Delta_{F,\mathsf{SAMP}_{\tau}(\mathcal{D}_{j,v}),\mathsf{SAMP}_{\hat{\tau}}(\mathcal{D})}(\varepsilon) &\leq \max_{w \in W} \int_{S'} \mu_{j,v} \mathcal{K}_{\tau} \mathcal{K}_{F}(a) - e^{\varepsilon} \ \mu_{j,w} \mathcal{K}_{\tau} \mathcal{K}_{F}(a) \ \mathbf{d}a \\ &\leq \max_{w \in W} \ \max_{S \subseteq A} \ \int_{S} \mu_{j,v} \mathcal{K}_{\tau} \mathcal{K}_{F}(a) - e^{\varepsilon} \ \mu_{j,w} \mathcal{K}_{\tau} \mathcal{K}_{F}(a) \ \mathbf{d}a \\ &= \max_{w \in W} \ \Delta_{F,\mathsf{SAMP}_{\tau}(\mathcal{D}_{j,v}),\mathsf{SAMP}_{\tau}(\mathcal{D}_{j,w})}(\varepsilon). \end{split}$$

# 4 Privacy Amplification by Subsampling

Serveral papers have discussed how subsampling can amplify privacy parameters in the DP model (see among others [14,15,1]). Where the effect on privacy for a single "worst case" database is considered. On the contrary, SP considers a distribution over possible databases, thus the interaction between the sampling distribution and the database distribution has to be analyzed. It turns out that sampling techniques where entries can appear multiple times are technically more difficult to handle because of possible dependencies in the sample.

### 4.1 Sampling without Replacement

Sampling without replacement has been investigated for differential privacy in several papers, among others [14,15,1]. By extending the coupling technique of Balle et al. to our kernel characterization of statistical privacy we can show

**Theorem 1.** Let  $\mu$  be a distribution for databases of size n, F a query and  $\mathcal{T}_{n,m}$  sampling without replacement with sample size m. Then  $F \circ SAMP_{\mathcal{T}_{n,m}}$  achieves  $(\log (1 + \frac{m}{n} (e^{\varepsilon} - 1)), \frac{m}{n} \operatorname{SPC}_{F,\mu,\mathcal{T}_{n,m}}(\varepsilon))$ -statistical privacy for  $\mu$ .

*Proof.* Let  $\varepsilon' = \log(1 + \frac{m}{n}(e^{\varepsilon} - 1))$  and  $v, w \in W$ . The total variation of  $\mu_{j,v}\mathcal{K}_{\mathcal{T}}$  and  $\mu_{j,w}\mathcal{K}_{\mathcal{T}}$  is at most m/n - the probability that the sensitive entry j is drawn. Thus there exist a maximal coupling with parameter m/n such that

$$\mu_{j,v}\mathcal{K}_{\mathcal{T}} = \left(1 - \frac{m}{n}\right) \mu_{j,v}\mathcal{K}_{\mathcal{T}_{j,-}} + \frac{m}{n} \mu_{j,v}\mathcal{K}_{\mathcal{T}_{j,+}}$$
$$\mu_{j,w}\mathcal{K}_{\mathcal{T}} = \left(1 - \frac{m}{n}\right) \mu_{j,v}\mathcal{K}_{\mathcal{T}_{j,-}} + \frac{m}{n} \mu_{j,w}\mathcal{K}_{\mathcal{T}_{j,+}} .$$

Because of the advanced joint convexity property it holds

$$\Delta_{F,\mu_{j,v}\mathcal{K}_{\mathcal{T}},\mu_{j,w}\mathcal{K}_{\mathcal{T}}}(\varepsilon') \\ \leq \frac{m}{n} \left( (1 - e^{\varepsilon' - \varepsilon}) \Delta_{F,\mu_{j,v}\mathcal{K}_{\mathcal{T}_{j,+}},\mu_{j,v}\mathcal{K}_{\mathcal{T}_{j,-}}}(\varepsilon) + e^{\varepsilon' - \varepsilon} \Delta_{F,\mu_{j,v}\mathcal{K}_{\mathcal{T}_{j,+}},\mu_{j,w}\mathcal{K}_{\mathcal{T}_{j,+}}}(\varepsilon) \right).$$

By definition we have that

$$\Delta_{F,\mu_{j,v}\mathcal{K}_{\mathcal{T}_{j,+}},\mu_{j,w}\mathcal{K}_{\mathcal{T}_{j,+}}}(\varepsilon) \leq \operatorname{SPC}^{j}_{F,\mathcal{D},\mathcal{T}_{j,+}}(\varepsilon) = \operatorname{SPC}^{j}_{F,\mathcal{D},\mathcal{T}}(\varepsilon) .$$

Consider a coupling  $\nu_{\mathcal{T}_{j,+},\mathcal{T}_{j,-}}$  of  $\mathcal{T}_{j,+}$  and  $\mathcal{T}_{j,-}$  which matches all subsamples equal except where  $\mathcal{T}_{j,+}$  places the sensitive entry. At these locations  $\mathcal{T}_{j,-}$  selects an entry not drawn yet uniformly distributed. Thus for all  $(\tau_+, \tau_-) \in$ supp  $\nu_{\mathcal{T}_{j,+},\mathcal{T}_{j,-}}$  one can apply Lemma 4 to the sampled databases. Doing this after using Lemma 2 we can bound the privacy curve as follows:

$$\begin{split} \Delta_{F,\mu_{j,v}\mathcal{K}_{\mathcal{T}_{j,+}},\mu_{j,v}\mathcal{K}_{\mathcal{T}_{j,-}}}(\varepsilon) &\leq \sum_{\tau_{+},\tau_{-}} \nu_{\mathcal{T}_{j,+},\mathcal{T}_{j,-}}(\tau_{+},\tau_{-}) \ \Delta_{F,\mu_{j,v}\mathcal{K}_{\tau_{+}},\mu_{j,v}\mathcal{K}_{\tau_{-}}}(\varepsilon) \\ &\leq \sum_{\tau_{+},\tau_{-}} \nu_{\mathcal{T}_{j,+},\mathcal{T}_{j,-}}(\tau_{+},\tau_{-}) \max_{w\in D} \ \Delta_{F,\mu_{j,v}\mathcal{K}_{\tau_{+}},\mu_{j,w}\mathcal{K}_{\tau_{+}}}(\varepsilon) \\ &\leq \operatorname{SPC}^{j}_{F,\mathcal{D},\mathcal{T}_{+}}(\varepsilon) = \operatorname{SPC}^{j}_{F,\mathcal{D},\mathcal{T}}(\varepsilon). \end{split}$$

This implies the claim

$$\Delta_{F,\mu_{j,v}\mathcal{K}_{\mathcal{T}},\mu_{j,w}\mathcal{K}_{\mathcal{T}}}(\varepsilon) \leq \frac{m}{n} \operatorname{SPC}^{j}_{F,\mathcal{D},T}(\varepsilon) \leq \frac{m}{n} \operatorname{SPC}_{F,\mathcal{D},\mathcal{T}}(\varepsilon) .$$

In case of identically and independently distributed entries Theorem 1 gives

**Corollary 1.** Let  $\overline{\mu}$  be the distribution of a database entry and  $\mu_n = \overline{\mu}^n$  and  $\mu_m = \overline{\mu}^m$  be the product distributions for databases of size n, resp. m. Then for a query F the sampling mechanism without replacement  $\mathcal{T}_{n,m}$  achieves  $(\log (1 + \frac{m}{n} (e^{\varepsilon} - 1)), \frac{m}{n} \Phi_{\mu_m,F}(\varepsilon))$ -statistical privacy with respect to  $\mu_n$ .

When does subsampling provide an improvement? For databases of size n the distribution  $\mu_n$  guarantees  $(\varepsilon, \Phi_{\mu_n,F}(e))$ -statistical privacy, while on a subset of size m one gets tuples  $(\hat{\varepsilon}, \Phi_{\mu_m,F}(\hat{\varepsilon}))$  for arbitrary  $\hat{\varepsilon} \geq 0$ . For small  $\hat{\varepsilon}$  the term  $\log(1 + \frac{m}{n}(e^{\hat{\varepsilon}} - 1))$  can be approximated by  $\lambda \hat{\varepsilon}$ , where  $\lambda = m/n$ . Thus the new privacy parameters are about  $(\lambda \hat{\varepsilon}, \lambda \Phi_{\mu_{\lambda n},F}(\hat{\varepsilon}))$ . This is a linear reduction in both parameters, but the distribution  $\mu_{\lambda n}$  on the smaller sample of size m provides less statistical privacy. To make the  $\varepsilon$ -parameter equal we set  $\hat{\varepsilon} = \lambda^{-1} \varepsilon$  and get for subsampling the pair  $(\varepsilon, \lambda \Phi_{\mu_{\lambda n},F}(\lambda^{-1} \varepsilon))$ . For a query F and distribution  $\mu$  subsampling increases privacy iff (approximately)

$$\Phi_{\mu,F}(\varepsilon) > \frac{m}{n} \operatorname{SPC}_{F,\mu,\mathcal{T}_{n,m}}(\log\left(1 + \frac{n}{m} \left(e^{\varepsilon} - 1\right)\right)) .$$
 (1)



**Fig. 1.**  $\delta$  values for property queries with p = 1/2 for different  $\varepsilon$  values. To ensure that the curves of  $\varepsilon = 1$ ,  $\varepsilon = 0.3$ , and  $\varepsilon = 0.1$  are better differentiated from each other, an upward correction of 0.001 was made for  $\varepsilon = 0.3$  and 0.002 for  $\varepsilon = 0.1$ .

Thus, the  $\delta$ -value decreases if the greater distributional divergence, when going from database size n to smaller size m, is compensated by the linear factor  $\lambda$  and the increase of the margin  $\varepsilon$  to  $\hat{\varepsilon} = \lambda^{-1} \varepsilon$ . Since the entropy of the distribution decreases linear in  $\lambda$ , applying a function F seems unlikely to give a larger entropy decrease. In addition the margin  $\hat{\varepsilon}$  grows linearly. Since

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there are no simple formulas for these relations one has to check the range of improvements for every  $\overline{\mu}$  and F. For large n, since  $\overline{\mu}^n$  converges to a normal distribution improvements should occur for a large range of  $\lambda$ .

As a typical example consider property queries which ask for the proportion of database entries that have a certain property that has a priori probability p(see[5]). Fig. 1 shows the dependency between n and  $\delta$  for different  $\varepsilon$ -values. For larger n the decrease of  $\delta$  becomes smaller.

By scaling the answer to a property query we get a counting query and since SP fulfills the postprocessing condition they are equivalent with respect to privacy. These counting queries are binomial distributed and for large n can be approximated by a normal distribution  $\mathcal{N}(n \ p, n \ p(1-p))$  if p is not close to 0 or 1. By considering the noise generated by the nonsensitive entries as additive noise in the DP sense, where counting queries have a sensitivity of 1, the DP privacy bound in Appendix A of [11] allows for the approximation

$$\Phi_{\mu_n,F}(\varepsilon) \approx 10/(n \ p(1-p) \ \varepsilon^2)$$

Thus for large m and n the inequality (1) is equal to

$$\begin{aligned}
\Phi_{\mu_n,F}(\varepsilon) &\geq \frac{m}{n} \Phi_{\mu_m,F}(\frac{n}{m} \varepsilon) \\
\iff & \Phi_{\mu_n,F}(\varepsilon) \geq \frac{m^2}{n^2} \frac{10}{(n \ p(1-p) \ (\varepsilon)^2)} \\
\iff & \Phi_{\mu_n,F}(\varepsilon) \geq \frac{m^2}{n^2} \Phi_{\mu_n,F}(\varepsilon) .
\end{aligned}$$

Since the last inequality always holds for m < n, we can conclude that the privacy of property queries is indeed amplified by subsampling. In Fig. 1 one can see that decreasing the database size n leads to a smaller increase of  $\delta$  (that means slope larger than -1) except for very small values of n. Fig. 2 gives a plot of the ratios of the exact values of the two sides in inequality (1). It shows that for smaller  $\varepsilon$  values the ratio is larger meaning subsampling is less effective, but still one gets an improvement.

However, subsampling always decreases the utility since when computing F on the whole database one gets the exact result while depending on the sampling rate  $\lambda$  and the function F the results on samples have some variance.

### 4.2 Poisson Sampling

Now consider Poisson sampling where every entry is drawn independently with the same probability  $\lambda$  – thus the sample size may vary. We can show two bounds for  $\delta$  and conjecture that the second one is never worse than the first one, but an analytic proof for this claim seems to be tedious.

**Theorem 2.** Let  $\mu$  be a database distribution, F a query and  $\mathcal{T}_{\lambda}$  Poisson sampling with rate  $\lambda$ . Then  $F \circ \text{SAMP}_{\mathcal{T}_{\lambda}}$  is  $(\varepsilon, \delta^{\star}(\varepsilon))$ -statistical private for

$$\delta^{\star}(\varepsilon) = \sum_{=0}^{n} \binom{n}{m} \lambda^{m} (1-\lambda)^{n-m} \frac{m}{n} \operatorname{SPC}_{F,\mu,\mathcal{T}_{n,m}} \left( \log \left( 1 + \frac{n}{m} \left( e^{\varepsilon} - 1 \right) \right) \right).$$



Fig. 2. Ratio of the  $\delta$  parameter with, resp. without subsampling given a database of size n = 1000 for property queries with p = 1/2 and different sampling rates  $\lambda$ . Here  $\varepsilon$  takes the values 0.1, 0.075, 0.05 and 0.025.

*Proof.* Let  $v, w \in W$  and  $\varepsilon > 0$ . Recognize, that the sample distributions can be broken down by size  $m \in [0:n]$  for  $u \in \{v, w\}$ , it holds

$$\mu_{j,u}\mathcal{K}_{\mathcal{T}} = \sum_{m=0}^{n} \binom{n}{m} \lambda^{m} (1-\lambda)^{n-m} \mu_{j,v} \mathcal{K}_{\mathcal{T}^{m}} , \qquad (2)$$

(3)

This allows for the following bound for any  $\varepsilon > 0$ 

$$\Delta_{F,\mu_{j,v}\mathcal{K}_{\mathcal{T}},\mu_{j,w}\mathcal{K}_{\mathcal{T}}}(\varepsilon) \tag{4}$$

$$= \int_{A} \max(0, \ \mu_{j,v} \mathcal{K}_{\mathcal{T}} \mathcal{K}_{F}(a) - e^{\varepsilon} \ \mu_{j,w} \mathcal{K}_{\mathcal{T}} \mathcal{K}_{F}(a)) \ \mathbf{d}a$$
(5)

$$\leq \sum_{m=0}^{n} {n \choose m} \lambda^{m} (1-\lambda)^{n-m} \int_{A} \max(0, \ \mu_{j,v} \mathcal{K}_{\mathcal{T}^{m}} \mathcal{K}_{F}(a) - e^{\varepsilon} \ \mu_{j,w} \mathcal{K}_{\mathcal{T}^{m}} \mathcal{K}_{F}(a)) \ \mathbf{d}a$$
(6)

$$=\sum_{m=0}^{n} \binom{n}{m} \lambda^{m} (1-\lambda)^{n-m} \Delta_{F,\mu_{j,v} \mathcal{K}_{\mathcal{T}^{m}},\mu_{j,w} \mathcal{K}_{\mathcal{T}^{m}}}(\varepsilon)$$
(7)

Since  $\mathcal{T}^m$  now draws a sample of fixed size m where every entry has the same probability to be drawn exactly once this realizes sampling without replacement for a sample size of m thus we can apply Theorem 1 which bounds  $\Delta_{F,\mu_{j,v}\mathcal{K}_{\mathcal{T}},\mu_{j,w}\mathcal{K}_{\mathcal{T}}}(\varepsilon)$  by

$$\sum_{m=0}^{n} \binom{n}{m} \lambda^m (1-\lambda)^{n-m} \frac{m}{n} \operatorname{SPC}_{F,\mu,\mathcal{T}_{n,m}} \left( \log \left( 1 + \frac{n}{m} \left( e^{\varepsilon} - 1 \right) \right) \right).$$

In the case of i.i.d entries we get:

**Corollary 2.** Let  $\overline{\mu}$  be the distribution of a database entry and  $\mu_m = \overline{\mu}^m$  the product distributions for a database of size m. Then for a query F Poisson sampling with parameter  $\lambda$  and

$$\delta^{\star}(\varepsilon) = \sum_{m=0}^{n} \binom{n}{m} \lambda^{m} (1-\lambda)^{n-m} \frac{m}{n} \Delta_{F,\mu_{m}} \left( \log \left( 1 + \frac{n}{m} \left( e^{\varepsilon} - 1 \right) \right) \right)$$

 $F \circ \text{SAMP}_{\mathcal{T}_{\lambda}}$  achieves  $(\varepsilon, \delta^{\star}(\varepsilon))$ -statistical privacy.

Thus, Poisson sampling with parameter  $\lambda$  improves privacy if

$$\Phi_{\mu,F}(\varepsilon) > \sum_{m=0}^{n} \binom{n}{m} \lambda^m (1-\lambda)^{n-m} \frac{m}{n} \operatorname{SPC}_{F,\mu,T^m}(\log\left(1+\lambda\left(e^{\varepsilon}-1\right)\right)) .$$

Fig. 3 visualizes the effect of Poisson sampling for property queries. As for sampling without replacement the improvement by Poisson sampling decreases when  $\varepsilon$  gets smaller.



Fig. 3. Ratio of the  $\delta$  parameter for Poisson subsampling given a database of size n = 100 for property queries with p = 1/2 and different sampling rates  $\lambda$ . Here  $\varepsilon$  takes the values 0.1, 0.075, 0.05, 0.025.

#### 4.3 Sampling with Replacement

Sampling with replacement turns out to be technically most difficult to analyse with respect to statistical privacy. The main problem here is the dependency when a critical entry is drawn more than once. For this reason we need the technical condition that the distribution is F-samplable. Let  $\mathcal{T}_k$  denote sampling where the sensitive entry is drawn k times.

**Theorem 3.** Let F be a monotone query,  $\mu$  a F-samplable distribution and  $\mathcal{T}_{n,m}^+$  sampling with replacement genearating a sample of size m from a set of

size n. Then  $F \circ \text{SAMP}_{\mathcal{T}_{n,m}^+}$  achieves

$$(\log(1+(1-(1-1/n)^m)(e^{\varepsilon}-1)), \ \frac{n}{m} \ \sum_{k=1}^m \binom{n}{k} (1/n)^k (1-1/n)^{(m-k)} \ \mathrm{SPC}_{F,\mathcal{D},\mathcal{T}_k}(\varepsilon))$$

statistical privacy.

Proof. Let  $\lambda = m/n$  and  $\varepsilon' = \log(1 + \lambda(e^{\varepsilon} - 1)), j \in [1:n]$  and  $v, w \in W$ . To ease the notation we simply write  $\mathcal{T}$  for  $\mathcal{T}^+_{n,m}$ . As for sampling without replacement we consider a maximal coupling matching the distributions where the sensitive entry is drawn. Here the total variation of  $\mu_{j,v}\mathcal{K}_{\mathcal{T}}$  and  $\mu_{j,w}\mathcal{K}_{\mathcal{T}}$  is at most  $\lambda = 1 - (1 - 1/n)^m$  the probability to not draw the sensitive element. Using the advanced joint convexity property we bound  $\Delta_{F,\mu_{j,v}\mathcal{K}_{\mathcal{T}},\mu_{j,w}\mathcal{K}_{\mathcal{T}}}(\varepsilon')$  by

$$\lambda \left( (1 - e^{\varepsilon' - \varepsilon}) \,\Delta_{F,\mu_{j,v} \mathcal{K}_{\mathcal{T}_{j,+}},\mu_{j,v} \mathcal{K}_{\mathcal{T}_{j,-}}}(\varepsilon) \,+\, e^{\varepsilon' - \varepsilon} \,\Delta_{F,\mu_{j,v} \mathcal{K}_{\mathcal{T}_{j,+}},\mu_{j,w} \mathcal{K}_{\mathcal{T}_{j,+}}}(\varepsilon) \right). \tag{8}$$

Let us first bound the second summand of (8) by applying Lemma 2.

$$\Delta_{F,\mu_{j,v}\mathcal{K}\mathcal{T}_{j,+},\mu_{j,w}\mathcal{K}\mathcal{T}_{j,+}}(\varepsilon) \leq \operatorname{SPC}^{j}_{F,\mathcal{D},\mathcal{T}_{j,+}}(\varepsilon) = \sum_{k\geq 0} \nu_{\mathcal{T}_{j,+}}(\mathfrak{C}_{j,k}) \operatorname{SPC}^{j}_{F,\mathcal{D},\mathcal{T}_{j,k}}(\varepsilon).$$

To bound the first summand consider the coupling  $\nu_{\mathcal{T}_{j,+},\mathcal{T}_{j,-}}$  that maps the templates  $\tau_+, \tau_-$  such that  $\tau_+ \approx_j \tau_-$ . This means they map the same entry if the entry is not the sensitive one. Else let I be the set of indices where  $(\tau_+)_k = j$ . Then for each  $z \in I$  the entry  $(\tau_-)_z$  is uniformly distributed on  $\{1, \ldots, n\} \setminus \{j\}$ and independent of the distributions of the entries  $(\tau_-)_v \ v \in I \setminus \{z\}$ . That coupling allows for the application of Lemma 5 after bounding with Lemma 3 thus

$$\begin{split} \Delta_{F,\mu_{j,v}\mathcal{K}_{\mathcal{T}_{j,+}},\mu_{j,v}\mathcal{K}_{\mathcal{T}_{j,-}}}(\varepsilon) &\leq \sum_{\tau_{+},\tau_{-}} \nu_{\mathcal{T}_{j,+},\mathcal{T}_{j,-}}(\tau_{+},\tau_{-}) \Delta_{F,\mu_{j,v}\mathcal{K}_{\tau_{+}},\mu_{j,v}\mathcal{K}_{\tau_{-}}}(\varepsilon) \\ &\leq \sum_{\tau_{+},\tau_{-}} \nu_{\mathcal{T}_{j,+},\mathcal{T}_{j,-}}(\tau_{+},\tau_{-}) \max_{w\in W} \Delta_{F,\mu_{j,v}\mathcal{K}_{\tau_{+}},\mu_{j,w}\mathcal{K}_{\tau_{+}}}(\varepsilon) \\ &\leq \operatorname{SPC}^{j}_{F,\mathcal{D},\mathcal{T}}(\varepsilon) = \sum_{k\geq 0} \nu_{\mathcal{T}_{j,+}}(\mathfrak{C}_{j,k}) \operatorname{SPC}^{j}_{F,\mathcal{D},\mathcal{T}_{j,k}}(\varepsilon) \,. \end{split}$$

Since the probability of drawing a specific entry k times is binomial distributed we can calculate  $\nu_{\mathcal{T}_{i,+}}(\mathfrak{C}_{i,k})$  giving

$$\begin{aligned} \Delta_{F,\mu_{j,v}\mathcal{K}_{\mathcal{T}},\mu_{j,w}\mathcal{K}_{\mathcal{T}}}(\varepsilon') &\leq \lambda \sum_{k=1}^{m} \binom{n}{k} (1/n)^{k} (1-1/n)^{(m-k)} \operatorname{SPC}_{F,\mathcal{D},\mathcal{T}_{j,k}}^{j}(\varepsilon) \\ &\leq \lambda \sum_{k=1}^{m} \binom{n}{k} (1/n)^{k} (1-1/n)^{(m-k)} \operatorname{SPC}_{F,\mathcal{D},\mathcal{T}_{k}}(\varepsilon) . \end{aligned}$$

As before the question arises when does sampling with replacement give an advantage, that means when holds

$$\Phi_{\mu,F}(\varepsilon) > \lambda \sum_{k=1}^{m} \binom{n}{k} (1/n)^{k} (1-1/n)^{(m-k)} \operatorname{SPC}_{F,\mu,\mathcal{T}_{k}}(\log\left(1+\frac{n}{m} \ (e^{\varepsilon}-1)\right)) \ .$$

This condition is significantly more complex than for sampling without replacement. However, we expect a similar behavior since the probability to draw the critical element several times decreases exponentially.

## 5 Trade-off Functions for Statistical Privacy

 $\varepsilon$ -divergence and  $(\varepsilon, \delta)$ -curves measure the amount of privacy that can be guaranteed. Dong et al. have proposed an alternative approach considering privacy as a hypothesis testing problem [9].

### **Definition 9. Trade-off Function**

For random variables  $X \sim \mu_x$  and  $Y \sim \mu_y$  on the same space A, the trade-off function  $T_{X,Y}: [0,1] \to [0,1]$  is defined by

$$T_{X,Y}(\alpha) := \inf_{S \subseteq A: \Pr[X \notin S] \le \alpha} \Pr[Y \in S]$$

Now the ability of an adversary to differentiate between two situations where a critical entry is either v or w is measured by such a function. This approach can be adapted to statistical privacy as well.

### Definition 10. T-Statistical Privacy (T-SP)

For a distribution  $\mu$ , a query F and a trade-off function T, a mechanism M achieves **T-statistical privacy** if for  $\mathcal{D} \sim \mu$ , all  $j \in [1:n]$  and  $w, v \in W$  holds

$$T_{M(F,\mathcal{D}_{j,w}),M(F,\mathcal{D}_{j,v})} \geq T$$
.

To apply this notion certain operations on trade-off functions are needed:

**Convex Conjugate:**  $T^{\star}(y) := \sup_{x} y \cdot x - T(x)$ 

**Inverse:**  $T^{-1}(\alpha) := \inf \{ x \in [0,1] \mid T(x) \le \alpha \}$ 

**Sample:**  $T_p(x) := p T(x) + (1-p)(1-x)$ 

*p*-sampling-operator:  $\Psi_p(T) := \min \left( \left\{ T_p, T_p^{-1} \right\}^* \right)^*$ 

The analysis that links  $(\varepsilon, \delta)$ -DP to trade-off functions can be adapted to statistical privacy.

#### Theorem 4.

For a distribution  $\mu$  and a query F, a mechanism M achieves T-statistical privacy for the trade-off function defined by

$$T(\alpha) := \sup_{\varepsilon \ge 0} \max \{ 0, \ 1 - \Phi_{\mu,F,M}(\varepsilon) - e^{\varepsilon} \alpha, \ e^{-\varepsilon} (1 - \Phi_{\mu,F,M}(\varepsilon) - \alpha) \}.$$

Furthermore, if for a trade-off function T one achieves T-statistical privacy then in the standard notion one achieves  $(\varepsilon, 1 + T^*(-e^{\varepsilon}))$ -statistical privacy.

In case of subsampling the situation is more complex since the size of the database has a larger influence for statistical privacy. For differential privacy it is often assumed that the sensitivity is independent of the database size, which seems questionable, for example in case of property queries. Under this

assumption the effect of subsampling has been analysed precisely. If a mechanism M achieves  $(\varepsilon, \delta)$ -DP then  $M \circ \text{SAMP}_{\mathcal{T}}$  achieves  $(\log (1 + \lambda(e^{\varepsilon} - 1)), \lambda \delta)$ -DP for a sampling technique T without replacement and sampling rate  $\lambda$  [1].

For statistical privacy the uncertainty of an adversary is depends on the size of the database. Thus the techniques used in the the proofs for the T-DP subsampling theorem need to be adjusted to utilize the parameters achieved when considering a smaller database.

**Theorem 5.** Le  $\overline{\mu}$  be a distribution of entries,  $\mu_n = \overline{\mu}^n$  and  $\mu_m = \overline{\mu}^m$  be the corresponding product distributions for databases of size n, resp. m and  $\mathcal{T}_{n,m}$  sampling without replacement. If for a trade-off function T one can achieve T-statistical privacy for  $\mu_m$  and F then  $F \circ \text{SAMP}_{\mathcal{T}_{n,m}}$  achieves  $\Psi_{\frac{m}{n}}(T)$ -statistical privacy for  $\mu_n$ .

*Proof.* The proof uses the machinery of [9], but instead of the sampling result for DP we use corollary 1 together with Theorem 4. F is  $(\varepsilon, 1 + T_m^{\star}(-e^{\varepsilon}))$ -SP under  $\mu_m$ . Thus  $F \circ \text{SAMP}_{\mathcal{T}_{n,m}}$  is  $(\log(1 + \frac{m}{n}(e^{\varepsilon} - 1)), \frac{m}{n}(1 + T^{\star}(-e^{\varepsilon})))$ -SP for  $\mu_n$ .

# 6 Conclusion

This paper has shows, that in the statistical adversarial model sampling can further increase privacy significantly. To prove this result we have developed methods to analyze the complex combination of sampling techniques and database distributions. Compared to a worst case analysis in DP, in SP one has to carefully keep track of all generated mixture distributions and the dependencies in case of sampling with replacement. Our results are comparable to the results for DP.

We have applied our framework to property queries to get quantitative data. As a next step one could consider other types of queries in the statistical setting. The privacy utility tradeoff should be analyzed for sampling in more detail. Furthermore, the effect of subsampling when composing queries needs to be investigated.

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# A Markov Kernels and Couplings

Let us repeat some basic notions concerning Markov Kernels and Couplings.

### Definition 11. Markov Kernel

Let  $(X, \mathcal{X})$  and  $(Y, \mathcal{Y})$  be measurable spaces. A function  $\mathcal{K} : \mathcal{Y} \times X \to [0, 1]$  is called a *Markov kernel* from  $(X, \mathcal{X})$  to  $(Y, \mathcal{Y})$  if the following properties are fulfilled:

- 1. For all fixed  $\hat{Y} \in \mathcal{Y}$ , the map  $x \mapsto \mathcal{K}(\hat{Y}, x)$  is  $\mathcal{X}$ - $\mathcal{B}([0, 1])$ -measurable, where  $\mathcal{B}([0, 1])$  denotes the Borel  $\sigma$ -algebra.
- 2. For all fixed  $x \in X$ , the map  $\hat{Y} \mapsto \mathcal{K}(\hat{Y}, x)$  is a probability measure on  $(Y, \mathcal{Y})$ .

A function  $\mathfrak{K}: Y \times X \to [0, \infty]$  is called the kernel density of  $\mathcal{K}$  with respect to a measure  $\mu$  on Y if for all  $\hat{Y} \in \mathcal{Y}$  and  $x \in X$  it holds

$$\mathcal{K}(\hat{Y},x) \;=\; \int_{\hat{Y}} \mathfrak{K}(y,x) \; \mathbf{d} \mu(y).$$

The Markov kernel of a measurable deterministic function f is defined by its kernel density  $\Re_f(x,y) = \hat{\delta}_{f(x)}(y)$  with respect to the Lebesgue measure. Thus, its Markov kernel equals

$$\mathcal{K}(\hat{Y}, x) = \mathbf{1}_{\hat{Y}}(f(x)) = \mathbf{1}_{f^{-1}(\hat{Y})}(x) .$$

Furthermore, for every function mapping elements of a set X to random variables on a set Y there exists a Markov kernel describing its stochastic behavior, in particular for privacy techniques applied to a database query F.

For a distribution  $\mu$  on  $(X, \mathcal{X})$ , a kernel  $\mathcal{K} : \mathcal{Y} \times X \to [0, 1]$  from  $(X, \mathcal{X})$  to  $(Y, \mathcal{Y})$  acts on  $\mu$  by

$$\mu \mathcal{K}(\hat{Y}) := \int_X \mathcal{K}(\hat{Y}, x) \, \mathbf{d}\mu(x).$$

Thus,  $\mu \mathcal{K}$  is a distribution on  $(Y, \mathcal{Y})$ . For Markov kernels  $\mathcal{K}_1$  from  $(X, \mathcal{X})$  to  $(Y, \mathcal{Y})$  and  $\mathcal{K}_2$  from  $(Y, \mathcal{Y})$  to  $(Z, \mathcal{Z})$  the composition  $\mathcal{K}_1 \mathcal{K}_2 : \mathcal{Z} \times X \to [0, 1]$  is a Markov kernel from  $(X, \mathcal{X})$  to  $(Z, \mathcal{Z})$ .

Couplings are a technique to compare two distributions  $\mu$  and  $\hat{\mu}$ . These are 2dimensional distributions whose marginal distributions are equal to  $\mu$  and  $\hat{\mu}$ . Of interest are couplings that maximizes the similarity of  $\mu$  and  $\hat{\mu}$ , called a maximal coupling.

### **Definition 12.** Maximal Couplings

Given two distributions  $\mu$  and  $\hat{\mu}$  whose total variation equals  $\lambda$ , a maximal coupling is a triple of distributions  $\mu^0, \mu^1, \hat{\mu}^1$  such that

$$\mu = (1 - \lambda) \mu^0 + \lambda \mu^1$$
$$\hat{\mu} = (1 - \lambda) \mu^0 + \lambda \hat{\mu}^1$$

 $\mu^0$  is the part where both distributions coincide and  $\mu^1, \hat{\mu}^1$  where they differ. A useful tool to analyze privacy is the *advanced joint convexity* property of the  $\alpha$ -divergence (see [1]). In our setting this corresponds to the following:

#### Lemma 6. Advanced Joint Convexity

For distributions  $\mu, \hat{\mu}$  with total variation  $\lambda$  and a query F let  $\mu^0, \mu^1, \hat{\mu}^1$  be a maximal coupling. Then for  $\varepsilon' = \log(1 + \lambda(e^{\varepsilon} - 1)) \approx \lambda \varepsilon$  it holds

$$\Delta_{F,\mu,\hat{\mu}}(\varepsilon') \leq \lambda \left( (1 - e^{\varepsilon' - \varepsilon}) \ \Delta_{F,\mu_1,\mu_0}(\varepsilon) + e^{\varepsilon' - \varepsilon} \ \Delta_{F,\mu_1,\hat{\mu}_1}(\varepsilon) \right)$$

# B Class of *F*-samplable distributions

There are many distributions commonly appearing in real world data making it necessary to try to analyze a wide array of different queries and database distributions. The exponential family is a set of distributions where the density function  $\mu$ , which depends on some parameter vector  $p \in \Pi$ , can be expressed using functions  $T: A \to \mathbb{R}^k, g: \Pi \to \mathbb{R}, \xi: \Pi \to \mathbb{R}^k$  and non-negative  $h: A \to \mathbb{R}$ as

$$\mu(x) := g(p)h(x)e^{\xi(p)T(x)^T}$$

**Theorem 6.** Let F be a query F and  $\mathcal{D}$  a database. When for all  $v, w \in W, j \in [1:n]$  and  $\tau in \mathfrak{C}^n$  the distributions of  $F(\mathsf{SAMP}_\tau \mathcal{D}_{j,v})$  and  $F(\mathsf{SAMP}_\tau \mathcal{D}_{j,w})$  are part of the exponential family with parameters  $p, p' \in \Pi$  such that

$$\sum_{i=1}^{k} T_i(x)(\xi_i(p) - \xi_i(p'))$$

is a polynomial of degree one or less. Then F if F-samplable with respect to  $\mathcal{D}$ .

*Proof.* One needs to asses the set  $S \subseteq A$  that maximizes the integral

$$\int_{S} \mu_{j,v} \mathcal{K}_{\tau} \mathcal{K}_{F}(x) - e^{\varepsilon} \ \mu_{j,w} \mathcal{K}_{\tau} \mathcal{K}_{F}(x) \quad \mathbf{d}x$$

for all  $\varepsilon > 0$ . Thus by evaluating the zero of  $\mu_{j,v} \mathcal{K}_{\tau} \mathcal{K}_F(x) - e^{\varepsilon} \mu_{j,w} \mathcal{K}_{\tau} \mathcal{K}_F(x)$ the structure of S can be determined. Since the evaluated densities are in the exponential family this amounts to

$$0 = g(p)h(x)e^{\xi(p)T(x)^{T}} - g(p')e^{\varepsilon}h(x)e^{\xi(p')T(x)^{T}}$$
  

$$\Leftrightarrow 0 = \frac{g(p)e^{\sum_{i=0}^{k}\xi_{i}(p)T_{i}(x)}}{g(p')e^{\varepsilon}e^{\sum_{i=0}^{k}\xi_{i}(p')T_{i}(x)}} - 1$$
  

$$\Leftrightarrow \log(1) = \log\left(\frac{g(p)}{g(p')e^{\varepsilon}}\prod_{i=0}^{k}\frac{e^{\xi_{i}(p)T_{i}(x)}}{e^{\xi_{i}(p')T_{i}(x)}}\right)$$
  

$$\Leftrightarrow 0 = \log\left(\frac{g(p)}{g(p')e^{\varepsilon}}\right) + \sum_{i=0}^{k}T_{i}(x)(\xi_{i}(p) - \xi_{i}(p')).$$

When the last sum is a polynomial of degree zero the integrant is a constant c allowing for  $S = \emptyset$  depending of the sign of c. If the sum is a polynomial of degree one the last equality has one solution  $x^*$  for x and thus  $S = (-\infty, x^*]$  or  $S = [x^*, \infty)$ .

Since the way in which the final distribution of the query answer is generated this requirement needs to be evaluated for every case. As a general Rule it holds often if the rate or variace parameters are fixed. This can be observed for the normal or Gamma distributions and holds as well for discrete members of the exponential family like Binomial, Geometic or Poisson distributions.

Sum Queries Sum queries,  $g_{\alpha} : \mathbb{R}^n \to \mathbb{R}$  with  $g_{\alpha}(x) = \sum_{i=0}^n \alpha_i x_i$  for some fixed  $\alpha \in \mathbb{R}^n$ , are a common type of query considered in differential privacy.

**Theorem 7.** Let  $\mathcal{D}$  be a database and  $\alpha \in \mathbb{R}^N$  such that for all  $i \in [1:n]$  the random variables  $\alpha_i \mathcal{D}_i$  are identically distributed and they are all either Binomial, Poisson, Normal or Cauchy distributed then  $g_\alpha$  if F-samplable with respect to  $\mathcal{D}$ 

*Proof.* All distributions from the proposition are closed with respect to summation. And the first three are members of the exponential family where the problematic parameters are fixed. For Cauchy distributed answers the zero of the difference of the densities to  $Cau(a, \gamma)$  and  $Cau(a', \gamma)$  needs to be evaluated.

$$\begin{array}{rcl} 0 & = & \displaystyle \frac{1}{\pi} \left( \frac{\gamma}{(x-a')^2 + (\gamma)^2} - \frac{e^{\varepsilon}\gamma}{(x-a)^2 + (\gamma)^2} \right) \\ \Leftrightarrow 1 & = & \displaystyle \frac{(x-a)^2 + \gamma^2}{(x-a')^2 + \gamma^2} \end{array}$$

The second equality holds iff

$$(x-a)^2 = (x-a')^2$$

solving for  $x^* = ((a')^2 - a^2)/2(a' - a)$  and thus  $S = (-\infty, x^*]$  or  $S = [x^*, \infty)$ .

# C Poisson Sampling in Differential Privacy

Sampling for differential privacy is well studied (see among others [15,1,2]), the classic results state, that when sampling with rate  $\lambda$  without replacement a mechanisms fulfilling  $(\varepsilon, \delta)$ -DP becomes  $(\log (1 + \lambda(e^{\varepsilon} - 1)), \lambda \delta)$ -DP. In difference to the bounds for Poisson sampling in [1] we propose a new bound for the setting of

**Theorem 8.** Let F be a query and M some privacy technique achieving DP for a privacy curve  $\delta(\varepsilon)$ . Then the privacy technique  $M(F, SAMP_{\mathcal{T}}(\cdot))$  where  $\mathcal{T}$  is a Poisson sampling technique with sampling rate  $\lambda$  achieves DP for the privacy curve

$$\delta^{\star}(\varepsilon) = \sum_{m=0}^{n} ll^{m} (1-\lambda)^{n-m} \binom{n}{m} m/n\delta(\log\left(1+n/m(e^{\varepsilon}-1)\right)).$$

*Proof.* We follow our proof in 2 for our second bound. Here  $\mathcal{K}_{M,F}$  defines the Markov kernel corresponding to the application of  $M(F, \cdot)$ . Thus for any  $D, D' \in W^*$  with  $D \approx_c D'$  where  $\mu_D$  and  $\mu_{D'}$  define their point distributions we can represent their privacy curve  $d(\varepsilon) = \Delta_{(F,M),\mu_D\mathcal{K}_T,\mu_{D'}\mathcal{K}_T}(\varepsilon)$ . This allows for the same steps to split along the database size as in the SP proof leaving us with

$$d(\varepsilon) \leq \sum_{m=0}^{n} ll^{m} (1-\lambda)^{n-m} \binom{n}{m} \underbrace{\max_{S \subseteq A} \int_{S} \mu_{D} \mathcal{K}_{\mathcal{T}^{m}} \mathcal{K}_{M,F}(x) - e^{\varepsilon} \mu_{D'} \mathcal{K}_{\mathcal{T}^{m}} \mathcal{K}_{M,F}(x) \, \mathbf{d}x}_{(A)}}_{(A)}$$

As for SP one recognizes that (A) corresponds to the sampling without replacement technique applied to M and F with a sample size of m. Bounding (A) by choosing the maximum over all neighboring databases and applying the amplification bound for sampling without replacement on databases that differ in one entry stated in [2] we arrive at the Bound stated in the theorem.