LINEAR STABILITY ANALYSIS FOR A SYSTEM OF SINGULAR AMPLITUDE EQUATIONS ARISING IN BIOMORPHOLOGY

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Abstract. We study linear stability of exponential periodic solutions of a system of singular amplitude equations associated with convective Turing bifurcation in the presence of conservation laws, as arises in modern biomorphology models, binary fluids, and elsewhere. Consisting of a complex Ginzburg-Landau equation coupled with a singular convection-diffusion equation in "mean modes" associated with conservation laws, these were shown previously by the authors to admit a constant-coefficient linearized stability analysis as in the classical Ginzburg-Landau case- albeit now singular in wave amplitude ε- yielding useful *necessary* conditions for stability, both of the exponential functions as solutions of the amplitude equations, and of the associated periodic pattern solving the underlying PDE. Here, we show by a delicate two-parameter matrix perturbation analysis that (strict) satisfaction of these necessary conditions is also *sufficient* for diffusive stability in the sense of Schneider, yielding a corresponding result, and nonlinear stability, for the underlying PDE. Moreover, we show that they may be interpreted as stability along a non-normally hyperbolic slow manifold approximated by Darcy-type reduction, together with attraction along transverse mean modes, connecting with finite-time approximation theorems of Häcker-Schneider-Zimmerman.

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1. Introduction

In this paper, we carry out a linearized stability analysis for periodic traveling-wave solutions

$$(A, B)(\hat{x}, \hat{t}) = (A_0 e^{i(\kappa \hat{x} - \omega \hat{t})}, B_0), \tag{1.1}$$

$A_0, B_0 =$ constant, of a system of singular amplitude equations

$$\begin{aligned} A_{\hat{t}} &= a A_{\hat{x}\hat{x}} + bA + c|A|^2 A + dAB, \\ B_{\hat{t}} &= \varepsilon^{-1}(f B_{\hat{x}} + h|A|^2_{\hat{x}}) + e_B B_{\hat{x}\hat{x}} + \Re(g A \bar{A}_{\hat{x}})_{\hat{x}}, \end{aligned} \tag{mcGL}$$

with $A, a, b, c \in \mathbb{C}$, $B \in \mathbb{R}^m$, $d \in \mathbb{C}^{1\times m}$, $g, h \in \mathbb{R}^{m\times 1}$, $f, e_B \in \mathbb{R}^{m\times m}$, $\Re a, \Re b > 0$, $\Im \operatorname{spec}(f) = 0$, and $\Re \operatorname{spec}(e_B) > 0$, in the limit as $\varepsilon \to 0$.

The first example of such a system, and the striking phenomenon of singular convection in B, was derived via multi-scale expansion by Häcker, Schneider, and Zimmerman [HSZ] in the context of weakly unstable Bénard-Marangoni and thin-film flow, for which $d = 0$ and the equations partially decouple. The singular convection they observed is a surprising consequence of the interaction of conserved quantities and convective forces in the underlying pde. For, in the absence of conserved quantities, the relevant amplitude equations are the famous (nonsingular) complex Ginzburg-Landau equations [E, vH, KSM, M3] consisting of the A equation alone, whereas, in the absence of convection, the B equation becomes purely diffusive, and (mcGL) reduces to a nonsingular system discovered previously by Matthews and Cox [MC].

The more general, fully coupled version described above was derived in [WZ3] by multi-scale expansion as a set of amplitude equations formally governing weakly unstable behavior near Turing bifurcation of a general family of convection reaction diffusion system

$$u_t + f(u, \nu)_x - (B(u, \nu) u_x)_x = R(u, \nu) := \begin{pmatrix} R_1(u, \nu) \\ 0_m \end{pmatrix}, \qquad R_1 \text{ full rank.} \tag{1.2}$$

$u \in \mathbb{R}^n$, indexed by bifurcation parameter ν centered about 0, in the presence of conservation laws: that is, for R of co-rank $0 < m \leq n$. As described in [WZ1, WZ3], the main motivation for this study was from modern biomorphology models incorporating mechanical/hydrodynamical effects, in particular vasculogenesis models as in [AGS, SBP].

In particular, the periodic solutions (1.1) of (mcGL) correspond to approximate solutions

$$U^\varepsilon(\xi, \hat{x}, \hat{t}) = \frac{1}{2} A(\hat{x}, \hat{t}) e^{i\xi} r + c.c. + \varepsilon^2 (B(\hat{x}, \hat{t}) + c.c. + H.O.T. \tag{1.3}$$

of (1.2), where $\xi = k_*(x - d_* t)$, $\hat{x} = \varepsilon(x - (d_* + \delta)t)$, and $\hat{t} = \varepsilon^2 t$ for d_*, δ real numbers determined by formal expansion, *c.c.* denotes complex conjugate, and *H.O.T* denotes higher-order terms. Here, $\nu \sim \varepsilon^2$, while r corresponds to the bifurcating neutral direction for the Fourier symbol of the linearized equations about the constant state $u(x, t) \equiv U_0$ from which Turing bifurcation occurs. For relations of model parameters of (mcGL) to the form of (1.2), see [WZ3].

We refer to (mcGL) as *modified complex Ginzburg-Landau equations* by analogy to the standard complex Ginzburg-Landau equations playing a corresponding role for convective Turing bifurcation without conservation laws [M3, WZ1, WZ2].

Example 1.1. An example of (1.2) with $m = 1$, arising in biomorphology, is the variation on the hydrodynamic/chemotactic vasculogenesis models of [GAC, AGS]:[1]

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho u) &= \mu \Delta \rho, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) - \beta \rho \nabla c &= \nu \Delta u - \gamma \rho u, \\ \partial_t(c) &= D\Delta c + \alpha \rho - \tau^{-1} c; \end{aligned} \tag{1.4}$$

see also [LW]. Here, ρ and u are density and velocity of endothelial cells, P pressure, c chemo-attractant concentration, $\alpha > 0$ and $\beta > 0$ release and cell response rates, and $\tau > 0$ the half-life of the chemo-attractant. The $\gamma \geq 0$ and $\nu \geq 0$ terms model in different ways drag against the extracellular matrix. They are set to $\gamma > 0$, $\nu = 0$ in [AGS] and $\gamma = 0$, $\nu > 0$ in [GAC]; here, we take $\gamma, \nu > 0$. The term $\mu \Delta \rho$ is a nonphysical "artificial viscosity", set to $\mu = 0$ in [GAC, AGS] and here $\mu > 0$. In (1.1), the pressure is taken to be zero below some critical density; in [GAC] it is taken to be zero. Here, following [LW], we take $P(\rho) = A\rho^2$, or, more generally, $P = A\rho^p$, $p \geq 1$.

Existence. We recall briefly the existence theory for (mcGL), both for context/general interest, and to introduce a supercriticality condition that will be important in the later stability analysis. Substituting $|A| = A_0$, $B \equiv B_0$ into (mcGL) satisfies the second equation automatically and in the first equation gives a shifted complex Ginzburg-Landau equation

$$A_{\hat t} = aA_{\hat x \hat x} + \tilde b A + c|A|^2 A, \qquad \tilde b := b + dB_0 \tag{1.5}$$

for which periodic existence may be treated in standard fashion.

Specifically, substituting $A = A_0 e^{i(\omega \hat t + \kappa \hat x)}$ gives

$$A_0^2 = \frac{\Re(\tilde b) - \Re(a)\kappa^2}{-\Re(c)}. \tag{1.6}$$

Here, $\Re(a) > 0$ by parabolicity/well-posedness and $\Re(b) > 0$ by standard Turing assumptions/exchange of stability [WZ2]. We will assume also the supercriticality condition

$$\Re(c) < 0 \tag{1.7}$$

allowing (as is standard) existence of solutions with $\kappa = 0$, yielding finally the domain of existence

$$\kappa^2 < \kappa_{exist}^2 := \frac{\Re(\tilde b)}{\Re(a)} = \frac{\Re(b) + \Re(d)B_0}{\Re(a)}. \tag{1.8}$$

Finally, we assume

$$\Re\,\mathrm{spec}(izf + z^2 e_B) < 0 \text{ for nonzero } z \in \mathbb{R}, \tag{1.9}$$

as follows from a more detailed analysis of the Turing assumptions, namely, the assumption that the spectra of the linearized operator for (1.2) about the bifurcating constant solution be strictly negative for $\sigma \neq 0$ except for a single complex-conjugate pair with real part order ε^2. Taking account of the asymptotics relating (mcGL) to (1.2), this corresponds to the assumption on the linearization

$$\begin{aligned} A_{\hat t} &= aA_{\hat x \hat x} + bA + dAB, \\ B_{\hat t} &= \varepsilon^{-1} f B_{\hat x} + e_B B_{\hat x \hat x} \end{aligned} \tag{1.10}$$

[1] We note that the nonconservative form of term $\beta\rho\nabla c$, appearing in the second, *nonconservative* equation of (1.4), does not change the analysis, here or in [WZ1, WZ2, WZ3], in any appreciable way.

of (mcGL) about $(A, B) \equiv (0, B_0)$ that spectra be negative except for a single complex-conjugate pair with real part order 1. By inspection, the spectra consist of two copies of $\mathrm{spec}(b + a\sigma^2)$, $b > 0$, union with $\mathrm{spec}(\varepsilon^{-1} f\sigma + \sigma^2 e_b)$, whence (1.9) follows by setting $\sigma = \varepsilon^{-1} z$ to obtain

$$\mathrm{spec}(\varepsilon^{-1} f\sigma + \sigma^2 e_b) = \varepsilon^{-2}\, \mathrm{spec}(z f\sigma + z^2 e_b).$$

Remark 1.2. Note that (1.8) limits B_0 to range

$$B_0 \in \mathrm{sgn}\Re(d) \times (-\Re(b)/|\Re(d)|, +\infty). \tag{1.11}$$

In particular, the choice $B_0 = 0$ is always feasible. Contrarily, by the change of coordinate $B \to B - B_0$, one could take without loss of generality $B_0 = 0$ for any existing periodic wave, thereby changing b to $\tilde{b} > 0$.

Stability. As noted in [WZ3], just as for the classical complex Ginzburg-Landau equation, the linearized stability problem for (mcGL) about exponential waves (1.1) may, by the exponentially weighted change of coordinates $(A, B) = \Big((A_0 + u + iv)e^{i(\kappa\hat{x} - \omega\hat{t})}, B_0 + w)$, u, v, w real, be converted to *real, constant-coefficient* form

$$\begin{aligned} u_{\hat{t}} &= \Re(a)u_{\hat{x}\hat{x}} - \Im(a)v_{\hat{x}\hat{x}} - 2\kappa\big(\Im(a)u_{\hat{x}} + \Re(a)v_{\hat{x}}\big) + 2A_0^2\Re(c)u + A_0\Re(d)w,\\ v_{\hat{t}} &= \Im(a)u_{\hat{x}\hat{x}} + \Re(a)v_{\hat{x}\hat{x}} + 2\kappa\big(\Re(a)u_{\hat{x}} - \Im(a)v_{\hat{x}}\big) + 2A_0^2\Im(c)v + A_0\Im(d)w,\\ w_{\hat{t}} &= \varepsilon^{-1}(f w_{\hat{x}} + 2hA_0u_{\hat{x}}) + e_B w_{\hat{x}\hat{x}} + 2A_0\big(u_{\hat{x}}\Re(g) + v_{\hat{x}}\Im(g) + \kappa u\Im(g)\big)_{\hat{x}}, \end{aligned} \tag{1.12}$$

or, setting $\mathcal{U} = (u, v, w^T)^T$,

$$\mathcal{U}_t = \mathcal{L}\mathcal{U} := M_0\mathcal{U} + M_1(\varepsilon)\mathcal{U}_{\hat{x}} + M_2\mathcal{U}_{\hat{x}\hat{x}} \tag{1.13}$$

with M_j appropriately defined (see Section 2). Note that neither B_0 nor b appears.

The spectrum of the linearized operator $\mathcal{L}$ about the wave may thus be determined by linear algebraic computations, via the dispersion relation for (1.13), or

$$\hat{\lambda}(\sigma) \in \mathrm{spec}\, M(\varepsilon, \hat{\sigma}), \qquad M(\varepsilon, \hat{\sigma}) := M_0 + M_1(\varepsilon)\hat{\sigma} + M_2\hat{\sigma}^2 \tag{1.14}$$

with $\hat{\sigma}$ denoting Fourier wave number, a *global two-parameter matrix perturbation problem.*

Recall that spectral stability, necessary for linearized stability, corresponds to

$$\Re\hat{\lambda} \le 0 \text{ for } \hat{\lambda} = \hat{\lambda}(\hat{\sigma}) \in \mathrm{spec}(M(\varepsilon, \hat{\sigma})). \tag{1.15}$$

Meanwhile diffusive spectral stability in the sense of Schneider [S1, S2], sufficient for linearized and nonlinear stability, corresponds in this context to

$$\Re\hat{\lambda} \le c(\varepsilon)|\hat{\sigma}|^2/(1 + |\hat{\sigma}|^2) \text{ for } \hat{\lambda} = \hat{\lambda}(\hat{\sigma}) \in \mathrm{spec}(M(\varepsilon, \hat{\sigma})). \tag{1.16}$$

For general periodic waves of (1.2), Schneider's diffusive stability condition is defined in terms of Bloch-Floquet spectrum of the wave, as discussed in Section 6, below, with the Fourier wave number $\hat{\sigma}$ replaced by a Bloch-Floquet number. See Section 6.

Darcy reduction. A natural further reduction of (mcGL) [WZ3, HSZ] suggested by the singular structure of (mcGL) is to make the Ansatz $B + f^{-1}h|A|^2 = \tilde{B}_0$ for some fixed constant $\tilde{B}_0$. To relate the stability criteria of the Darcy reduction to the original model, we choose the constant $\tilde{B}_0$ by $\tilde{B}_0 = B_0 + f^{-1}h|A_0|^2$ for some fixed choice of periodic solution $(A_0e^{i(\kappa\hat{x}+\omega\hat{t})}, B_0)$ of (mcGL). Note then that $\hat{b}$ as in (1.18) is then a function of κ. Symbolically, we then obtain for $B = B(A)$

$$B(A) = B_0 + f^{-1}h|A_0|^2 - f^{-1}h|A|^2, \tag{1.17}$$

canceling the singular term in (mcGL)(ii), and giving for A the shifted complex Ginzburg-Landau equation

$$A_{\hat{t}} = aA_{\hat{x}\hat{x}} + \hat{b}A + \hat{c}|A|^2A, \text{ where } \hat{b} = b + d(B_0 + f^{-1}hA_0^2), \quad \hat{c} := c - df^{-1}h, \tag{1.18}$$

denoted the *Darcy* approximation by analogy to related approximations in hydrodynamics. See, e.g., [DJMR] for similar approximations in weakly nonlinear geometric optics and MHD. We note for later reference the relations (cf. (1.5))

$$\hat{b} = \tilde{b} + df^{-1}hA_0^2, \quad \hat{c} = c - df^{-1}h. \tag{1.19}$$

This does not represent an invariant subflow, but could be viewed, heuristically, as an approximate "slow manifold." We note that the corresponding fast flow

$$(B + f^{-1}h|A|^2)_{\hat{t}} = \varepsilon^{-1}\partial_{\hat{x}} f(B + f^{-1}h|A|^2)_{\hat{t}} + H.O.T.$$

linearized about a constant state has for spec(f) real, dispersion relation $\lambda(k) = i\,\mathrm{spec}\, f\varepsilon^{-1} + O(1)$ with real part of order 1 rather than ε^{-1}, so that the slow manifold is typically *non-normally hyperbolic*.

Noting further that periodic solutions, with $|A| = A_0$, $B = B_0$ constant, are contained entirely within the Darcy flow, we may think of these as lying within an exact slow manifold, with local dynamics about the periodic solutions given to first order by those of the Darcy flow (1.18). Hence, one may guess that *Darcy stability* of periodic waves, or stability as solutions of (1.18) is necessary- but not sufficient- for stability as solutions of (mcGL). Stability within the full model (mcGL) evidently involves also the question whether fast flow is attracting or repelling, which, by non-normal hyperbolicity, may be expected to be somewhat delicate.

1.1. **Previous work.**

The approximate solutions (1.3) may be shown in by now standard fashion to be close to exact periodic solutions of (1.2), with sharp error bounds; see [WZ3] for further details. Notably, this existence problem is *nonsingular*, as one can see by setting $|A|, B$ constant, thus eliminating the B-equation, and noting that the resulting dimensional solution count remains correct. Though we do not display it here, the analog of (mcGL) in the O(2)-symmetric reaction diffusion case is nonsingular in both A and B to begin with, and so the issue of singularity does not ever arise. For this reason we were able to treat in standard fashion not only existence but also stability completely for that case in [WZ3], using classical methods of Mielke and Schneider [M1, M2, S1, S2], showing under generic conditions that diffusive stability of periodic waves of (1.2) is equivalent to diffusive stability of the associated periodic waves (1.1) of (mcGL) in (1.3).

However, in the convective case modeled by (mcGL), the stability problem is not only singular, but the neutral eigenstructure of the Fourier symbol about the constant state U_0 features a Jordan block that would at first sight appear to prevent completely an analytic Taylor expansion about zero frequency as is the first step in the analysis of [M1, M2, S1, S2]. Using a key observation of [JZ1, JZ2, JNRZ], we were able to overcome this apparent obstacle and perform Taylor expansion on a ball of ε times smaller order than in the standard case, thus obtaining in [WZ3] *necessary but not sufficient* conditions for diffusive spectral stability in the sense of Schneider [S1, S1] corresponding to low-frequency diffusive stability of the associated periodic waves (1.1) of (mcGL) in (1.3).

We mention also the earlier results of [HSZ] of a different, *bounded-time approximation* type, for specific decoupled ($d = 0$) versions of (mcGL) arising in Bènard-Marangoni and thin film flow, in which they showed that a corresponding Darcy model for *localized* rather than periodic solutions (A, B) of (mcGL), with the modified relation $B = -f^{-1}h|A|^2$, accurately predicts bounded-time behavior, in the sense that localized solutions of the resulting Darcy model lie near exact solutions of the underlying PDE (1.2). In the decoupled case considered there, it was clear that this "approximate slow manifold" was attracting, suggesting at least heuristically formal link between bounded time approximation and time-asymptotic behavior. A natural question posed in [WZ3] was, in the fully coupled ($d \neq 0$) and perturbed periodic case considered there, to what extent the Darcy model (1.18) can shed light on *time-asymptotic behavior* as studied here.

1.2. **Main results.** The purpose of the present paper is to go beyond the restrictive regime imposed by Taylor expansion and complete a full linearized stability analysis, for both periodic solutions of the amplitude equations (mcGL) and the exact periodic solutions of (1.2) that the former approximate, yielding *sufficient as well as necessary conditions for diffusive stability.*

Main result 1. Our first main result is, under mild genericity assumptions, the derivation of $m+1$ simple necessary and sufficient conditions for diffusive spectral stability (2.5) of periodic solutions (1.1) of the amplitude equations (mcGL) for $0 < \varepsilon \le \varepsilon_0$ sufficiently small, where $m = \dim B$ is the number of mean modes. See Proposition 4.4 in the scalar case $m = 1$ and Proposition 5.4 in the vector case $m > 1$. Here, ε_0 may be chosen uniformly on compact parameter-sets for which the above-mentioned genericity conditions are satisfied.

The first condition, associated with the neutral translational mode is

$$\kappa^2 < \kappa_{stab}^2 := \frac{2\frac{\Re(\tilde b)}{\Re(c)}(\Im(a)\Im(\hat c) + \Re(a)\Re(\hat c))}{4\Re(a)^2(1+\hat q^2) + 2\frac{\Re(a)}{\Re(c)}\big(\Im(a)\Im(\hat c) + \Re(a)\Re(\hat c)\big)}, \tag{1.20}$$

deriving from the well-known Eckhaus condition [WZ3, Eqn. (2.13)]:

$$0 > \mu_t^0 := \frac{(2\kappa^2\Im\hat c^2\Re a^2 + A_0^2\Im a\Im\hat c\Re\hat c^2 + \Re a\Re\hat c^2(2\kappa^2\Re a + A_0^2\Re\hat c))2}{(A_0^2\Re\hat c^3)} \tag{1.21}$$

for the complex Ginzburg-Landau equation, with b, c replaced by $\hat b$, $\hat c$: that is, the translational stability condition for the Darcy approximation.

We note that (1.20) is *not* simply the κ boundary for the Darcy equation with b and c suitably replaced, since the relation $A_0^2 = \frac{\Re(\tilde b)-\Re(a)\kappa^2}{-\Re(c)}$ from (1.6) involves the original variable $\tilde b$ and c rather than $\hat b$ and $\hat c$. To put this a different way, from (1.19), we have that

$$\hat b = \tilde b + df^{-1}hA_0^2 \tag{1.22}$$

depends on A_0^2, so is not a fixed parameter. This subtle point is perhaps the main difference between the singular and the classical case. In the decoupled case $d = 0$, or for $\kappa = 0$, this distinction disappears.

We refer to condition (1.20) as the *generalized Eckhaus criterion*. The domain $\kappa^2 < \kappa_{stab}^2$ is nonempty under the Benjamin-Feir-Newell criterion

$$\Im a\Im\hat c\Re\hat b\Re\hat c + \Re a\Re\hat b\Re\hat c^2 > 0; \tag{BFN}$$

otherwise, diffusively stable waves do not exist. This corresponds to (1.21) with $\kappa = 0$, hence, unlike (1.20), is identical with the classical (BFN) condition with b, c replaced by $\hat b$, $\hat c$.

The remaining m conditions, associated with "conserved," or "mean" modes are both first-order and second-order. The first-order condition, automatic in the scalar case $m = 1$, is

$$\operatorname{spec}\left(f - h\tfrac{\Re(d)}{\Re(c)}\right) \text{ real.} \tag{1.23}$$

We make the additional nondegeneracy assumption

$$\operatorname{spec}\left(f - h\tfrac{\Re(d)}{\Re(c)}\right) \text{ distinct,} \tag{1.24}$$

denoting by ℓ_j and r_j associated left and right eigenvector pairs of $f - h\Re(d)/\Re(c)$. Then, the second-order conditions are

$$\mu_{c,j}^0 := \ell_j \frac{h\Re(d)}{2A_0^2\Re(c)}\Big(f - h\frac{\Re(d)}{\Re(c)}\Big)r_j < 0, \tag{1.25}$$

with no conditions on κ. We refer to these as *auxiliary Benjamin-Feir-Newell criteria* for the similar role they play to that of (BFN) in determination of stability.

The scalar case ($m = 1$). Specialized to a single conservation law, $m = 1$, as in the case of example system (1.4), the results simplify considerably, reducing to the pair of conditions

$$\Re(\mu_t) < 0 \text{ and } \Re(c) < \Re(\hat{c}) < 0. \tag{1.26}$$

Noting, as observed just above, that $\mu_t < 0$ agrees with the condition for translational stability of the Darcy approximation, while $\hat{c} < 0$ is necessary for supercriticality/fast mode stability of the same, we see that, apart from information obtainable from the Darcy "slow manifold" approximation, the only additional requirement is the condition $\Re(c) < \Re(\hat{c})$, which we will see in the computations later on is equivalent to stability of the transversal mean mode, or, heuristically, attraction of the Darcy slow manifold.

Main result 2. Our second main result is to connect stability of periodic solutions of the (mcGL) to stability of exact periodic solutions of (1.2) nearby the associated approximate solutions (1.3), the existence of which was established in [WZ3]. Namely, we show, under appropriate genericity assumptions, that for $0 < \varepsilon \leq \varepsilon_0$ sufficiently small, diffusive spectral stability in the sense of Schneider of exact periodic waves of (1.2) is equivalent to diffusive spectral stability (2.5) of approximating periodic waves (1.1) of the associated amplitude equations (mcGL) derived in [WZ3]; see Theorem 6.4. Here, again, ε_0 may be chosen uniformly on compact parameter-sets for which the required genericity conditions are satisfied. Thus, necessary and sufficient conditions for diffusive spectral stability of ε-amplitude exact periodic solutions of (1.2), with $0 < \varepsilon \leq \varepsilon_0$ sufficiently small, are given by the Eckhaus and auxiliary Benjamin-Feir-Newell conditions (1.20) and (1.23)-(1.25).

These complete the program of [WZ1, WZ2, WZ3, Wh]*, extending the theory of Eckhaus-Mielke-Schneider for classical Turing bifurcation to the case of Turing bifurcations with conservation laws*, such as occur in binary fluids and, most importantly for us, in biomorphogenesis models incorporating hydrodynamic and mechanical effects, the latter having received considerable recent interest.

Darcy approximation vs. the full model. In passing, we answer the question posed in [WZ3] of the relation between the Darcy approximation (1.18) and the full model (mcGL), showing that, indeed, Darcy stability is *necessary* (but not sufficient) for stability with respect to the full model of periodic solutions of (mcGL), (1.18). More precisely, we can see by expansion in ε with $\check{\sigma} := \varepsilon^{-1}\hat{\sigma}$ held fixed (carried out in Section 3) that Darcy stability is necessary for stability of dispersion relation (1.14) with wave numbers in intermediate frequency range $1/C \leq |\sigma| \leq C$, corresponding heuristically to behavior in the intermediate time range $0 < t_0 \leq t \leq 1$. By contrast, the Eckhaus conditions above are found by expanding in σ with ε fixed and bounded from zero. But, either by abstract matching arguments or by direct comparison, we see further that the leading order ultra-slow mode coefficient μ_t^0 obtained by either of these methods is identical. That is, the (bounded-time) Darcy and (time-asymptotic) Eckhaus slow modes agree to leading order, i.e., *the Darcy reduction behaves like an approximate slow manifold in both finite-time approximation and spectral sense*, similarly as shown in the classical (nonconservative) case by the analysis of Mielke and Schneider.

The fact that the order of limits does not affect the value of μ_t can be understood, more generally, from the fact proved in the course of the analysis of Section 4 (regions (ii)-(iii)), that *the slow mode μ_t is in fact jointly analytic* in ε, $\hat{\sigma}$ for ε, $|\hat{\sigma}|$ sufficiently small, similarly as in the classical case; see Remark 6.5. By contrast, the fast modes $\mu_{c,j}$, as noted in [WZ3] are typically analytic in $\hat{\sigma}$ only on the reduced range $|\hat{\sigma}| \ll \varepsilon$.

Concerning the values of $\mu_{c,j}$, the fact that $\Re(c) < 0$, $\mu_t^0 < 0$, and $\mu_{c,j}^0 < 0$ are sufficient for stability, together with the facts that the Darcy stability conditions $\Re(\hat{c})$, $\Re(\mu_t^0)$ are necessary for stability, implies that the former imply $\Re(\hat{c}) < 0$ also in the vector case $m > 1$, as was seen by direct computation in the scalar case $m = 1$. However, in the vector case, we do not see any easy way to show this directly.

1.3. **Discussion and open problems.** System (mcGL) generalizes amplitude equations derived for particular examples by Matthews-Cox [MC] in the O(2)-symmetric reaction diffusion case and Häcker-Schneider-Zimmermann [HSZ] in the SO(2)-symmetric reaction convection diffusion case. A major difference in the O(2) case is that the $\varepsilon^{-1}(fB_{\hat{x}} + h|A|^2_{\hat{x}})$ convection term does not appear in (mcGL)(ii), canceling due to reflective symmetry. This leaves a nonsingular reaction diffusion system in place of the coupled reaction diffusion/singular convection diffusion system of (mcGL), for which the analytic issues are essentially different. See [MC, S, WZ3] for further discussion.

The singular mean-mode convection studied here was first pointed out in [HSZ] in the context of weakly nonlinear asymptotics for Bénard-Marangoni convection and inclined-plane flow, with rather different motivations and results from those of the present analysis. More precisely, they sought to establish (1.3) together with (a localized version of) (1.18) as an infinite-dimensional approximate center manifold, lying ε^2-close to associated exact solutions in bounded time. To this end, they derived mode-filtered equations [HSZ, Eqs. (22)-(23) and (33)-(34] for exact solutions isolating neutrally growing (i.e., "slow") modes, corresponding under appropriate rescaling to a forced version of (mcGL), then showed by rigorous stability estimates that these remain ε^2 close to (1.3) under (localized) Darcy flow (1.18) for time interval $0 \le t \le C\varepsilon^{-2}$.

Note that the time-interval $0 \le t \le C\varepsilon^{-2}$ corresponds to a bounded time interval $0 \le \hat{t} \le C$ for (mcGL). Thus, the above stability estimates correspond to bounded-time, or well-posedness properties of (mcGL), rather than the asymptotic stability questions we consider here. Moreover, the class of localized (roughly, $H^s(\mathbb{R})$) solutions considered in [HSZ] does not include the periodic ones we study here. In addition, the solutions they construct are for "prepared" initial data, with B appropriately coupled to A, whereas we are concerned with stability with respect to general data and perturbations. On the other hand, the results of [HSZ] apply to a much larger class of solutions, whereas ours apply only to the periodic solutions (1.1) associated with Turing bifurcation.

A further major technical difference between the analysis of [HSZ] and that carried out here is that, for the particular models considered in [HSZ], the associated amplitude equations (mcGL) feature vanishing coupling coefficients $d = 0$ and simultaneously diagonal matrices f and e. Hence, the A equation (mcGL)(i) *decouples as a standard complex Ginzburg-Landau equation*, and the spectra of $\mathcal{L}$ in (1.13) decouples into the well-known spectra for the linearized complex Ginzburg-Landau equation plus that of the (already linear) decoupled B-equation $B_{\hat{t}} = \varepsilon^{-1} f B_{\hat{x}} + e_B B_{\hat{x}\hat{x}}$, or

$$\hat{\lambda}_j(\hat{\sigma}) = \varepsilon^{-1} i\hat{\sigma} f_j - \hat{\sigma}^2 (e_B)_j, \tag{1.27}$$

where f_j, $(e_B)_j$ denote the eigenvalues of f and e_B. Thus, time-asymptotic spectral stability is straightforward in that case, with the main technical issue being *stiffness*, or accounting of transient effects in B, with principal behavior given by the usual complex Ginzburg-Landau equation in A.

Indeed, the singular B-equation was not included in [HSZ] as part of the amplitude equations, but subsumed in the underlying analysis supporting the description of behavior by complex Ginzburg-Landau approximation, an aspect reflecting an important difference in point of view. For the general amplitude equations (mcGL), derived in [WZ3] in the context of biomorphogenesis models such as Murray-Oster and related equations for vasculogenesis [AGS, SBP], such a decoupling generically does not occur, and the determination of spectral stability changes from inspection as in (1.27), to the complicated singular two-parameter matrix perturbation problem studied here, in terms of ε and the Fourier (Bloch) number σ. At the same time, mean modes B are no longer necessarily decaying transients, but through coupling interactions may play an important role in dynamics. Thus, in the exposition of [WZ3], the mean modes B are "promoted" to elements of the formal amplitude equations, the latter becoming therefore *singular* with respect to ε.

It is truly remarkable that in the originally-motivating context of Turing bifurcation, the complicated singular two-parameter stability problem for (mcGL) in the end yields simple stability conditions (1.20) and (1.23)-(1.25) analogous to the classic Eckhaus and Benjamin-Feir-Newell stability conditions for periodic solutions of the complex Ginzburg-Landau equation. This both justifies the introduction of singular amplitude equations (mcGL) and sets the stage for systematic exploration of biological pattern processes like vasculogenesis to which they apply, starting from initiation at Turing bifurcation. *The latter program may be expected to play the same powerful role in exploration of global bifurcation in biomorphogenesis as has the corresponding program in the classical pattern-formation case arising in elasticity, reaction diffusion, and myriad other settings.*

Many biomorphology models used in applications feature partial parabolicity and mixed hyperbolic parabolic type. An important further open problem to extend our results for complete parabolicity to this more general domain, as been done for related settings in [JZN] and references therein. A second very important followup problem is to actually carry out the above-described program for specific biomorphology models, systematically determining associated Turing bifurcations and their stability; that is, studying the practical *initiation problem* for periodic pattern formation with conservation laws. In particular, it would be very interesting to carry out a full analysis for the example model discussed in Appendix A. A third, more speculative followup, as suggested in [WZ1, WZ3], is to study modulation of these patterns as a possible model for *emergent dynamics*; see [JNRZ, MetZ].

1.3.1. *Relations to thin-film flow.* More generally, our results in a sense also complete/complement the programs of [JNRZ], [JNRZ2, BJNRZ], and [BJZ1]. The analysis of [JNRZ] completely analyzes stability and asymptotic behavior given spectral diffusive stability condition of Schneider, while those of [JNRZ2, BJNRZ] verifies Schneider's condition in a certain degenerate small-amplitude limit arising in inclined shallow water flow. It was pointed out in [BJZ1] that nondegenerate Turing type bifurcations can also occur, as a complementary and in general perhaps more frequent case, and the determination of their stability, even partially or just heuristically, was cited as an important open problem. Moreover, it was observed that the numerics associated with spectral stability were quite delicate and computationally expensive for this problem.

Our present results both resolve this open problem in passing, giving simple conditions for diffusive spectral stability and justifying the heuristic approximation (mcGL), and explain the observed delicacy of numerics. For, the singularity ε^{-1} in (mcGL) makes this a stiff system in the $\varepsilon \to 0$ limit, for which the spectrum is inherently difficult to resolve. Nonetheless, the equations are theoretically well-posed [HSZ, WZ3]; the efficient resolution of the associated time-evolution problem is thus an interesting open problem. See [BLWZ] for work in this direction.

In regard to the related earlier work on thin film flow, it is worth noting that rescaling $\check{x} = \varepsilon\hat{x}$ (slow variable) removes ε^{-1} from (1.13), giving form

$$\mathcal{U}_t = \tilde{M}_1(\varepsilon)\mathcal{U}_{\check{x}} + M_0\mathcal{U} + \varepsilon^2 M_2 \mathcal{U}_{\check{x}\check{x}},$$

where $\tilde{M}_1(\varepsilon) = \tilde{M}_1^0 + \varepsilon\tilde{M}_1^1$, $\tilde{M}_1^j$ constant, that is, a singularly perturbed relaxation system with vanishing viscosity $\varepsilon^2\partial_{\check{x}}^2$, the form ultimately studied in the two-parameter matrix bifurcation analysis of Section 4. It is interesting that a similar relaxation structure was encountered in the studies [JNRZ2, BJNRZ] of a quite different bifurcation occurring in the small-amplitude limit for periodic waves in inclined shallow-water flow in the ultra-small frequency regime, and a reminiscent two-parameter bifurcation analysis successfully carried out. See Remark 4.5 for further discussion.

1.3.2. *The Darcy approximation revisited.* We emphasize that the bounded-time estimates of [HSZ], though they apply to more general types of solution, are not relevant to the special question of Turing patterns and their asymptotic stability, and so give essentially complementary information. We discuss this point further in Section 1.3.3. Likewise, the necessary conditions derived in [WZ3] were theoretical and not explicitly computed in the generic case considered here. As we shall see in Section 2, to go beyond establishing analyticity of neutral spectra and actually compute the first nonvanishing real parts of the associated Taylor series costs several further levels of expansion and substantial additional effort. However, the end result is quite simple, consisting of the conditions for stability within the formal Darcy approximation, plus stability of transverse mean modes: precisely as suggested by the heuristic picture of the Darcy model as an approximate slow manifold.

An interesting open problem related to the approximating manifold approach of [HSZ] is to extend their results to the generic case that coupling constant $d \neq 0$. As pointed out in [WZ3, §4.1], a corresponding ansatz in that case is $B = -h|A|^2/f$, canceling the singular term in (mcGL)(ii), and giving for A the modified complex Ginzburg-Landau equation (1.18) described in the introduction. This gives a similar approximation error in the full equations (mcGL) to that given in [HSZ] for the complex Ginzburg-Landau ansatz in the mode-filtered equations, suggesting that the latter should hold for the Darcy approximation in the general case $d \neq 0$ as well. Likewise there was established in [WZ3, §4.1] a bounded-time/well-posedness result for (mcGL) in class $H^s(\mathbb{R})$, verifying in the generic case $d \neq 0$ the second main ingredient in the analysis of [HSZ].

1.3.3. *Whitham modulation, amplitude equations, and the Darcy approximation.* We close with some comments of a general nature contrasting the various results obtained here and in [HSZ] and [JNRZ]. Here, we determine diffusive spectral stability in terms of the $m+1$ neutral (translational and conserved) modes of the linearized amplitude equations (mcGL) about periodic waves (1.1), and show that this is equivalent to diffusive stability of bifurcating periodic solutions of (1.2) for $\varepsilon > 0$ sufficiently small. The results of [JNRZ] give for *fixed* $\varepsilon > 0$ that low-frequency diffusive spectral stability of periodic solutions is equivalent to diffusive stability of the $(m+1)$-dimensional *Whitham modulation equations* for (1.2), which in turn implies linearized and nonlinear stability with respect to $H^s \cap L^1$ perturbations. The latter estimates, however, are quite ε-dependent, potentially blowing up as $\varepsilon \to 0$.

The Whitham modulation equations, likewise, are associated with the same neutral translational and conserved modes as considered in our spectral stability analysis. It is natural to conjecture that these should agree with the reduced equations that we obtain; at least the spectral expansions must agree to lowest order in appropriate domains of common validity. It would be very interesting to make this connection precise. Likewise, a difficult but extremely interesting open problem would be to give a description of behavior of perturbed Turing patterns like that of [JNRZ] but uniformly valid in ε.

These issues appear also related to the finite-time complex Ginzburg-Landau approximation of [HSZ], or, more generally, the Darcy approximation conjectured in Section 1.3.2, governed asymptotically by a single modulation equation describing translation, or "phase shift." In particular, one may ask whether the Darcy approximation could be not only valid for bounded time and vanishing ε, but also for small enough ε and time going to infinity, or both?

In this regard, we recall from [JNRZ] the related result that time-asymptotic behavior is dominated by phase shift precisely under a certain decoupling condition implying the absence of a Jordan block in the eigenstructure of neutral modes, corresponding to failure of our genericity condition (2.28).

$$\Im(d) = \Re(d)\Im(c)/\Re(c), \tag{1.28}$$

Thus, it is apparently not possible that the Darcy approximation, accounting for phase shift, can represent time-asymptotic behavior in the standard sense that remaining terms decay at faster time-asymptotic rate.

We conjecture, rather, that derivative of the phase and transverse, mean modes, decay at the same diffusive rate $t^{-1/2}$ in L^∞ (see [JNRZ]), but with coefficients of the latter of order ε, uniformly as $\varepsilon \to 0$. Note that this agrees with the intuition of Fourier modes with slow decay $e^{-\varepsilon^2 k^2 t}$, corresponding to a heat equation $u_t = \varepsilon^2 u_{xx}$ decay for L^1 initial data at rate $(\varepsilon^2 t)^{-1/2} = \varepsilon^{-1} t^{-1/1}$. If correct, this would yield that the Darcy approximation governs time-asymptotic behavior under general (not just "prepared" type) initial perturbations, a conclusion that would evidently be very useful if true. The resolution of this and the above related questions appear to be very interesting directions for future investigation.

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2. Necessity of Eckhaus conditions

We now begin the main task of the paper, analyzing the linear stability problem by reduction to constant coefficients followed by a 2-parameter matrix perturbation analysis. In this section, we carry out for fixed ε a Taylor expansion in σ about 0, determining necessary conditions for stability. In the following section, we examine the remaining regions of σ-ε space, showing that these conditions are in fact sufficient for stability as well.

For clarity, we first carry out the analysis completely in the case $m = 1$ of a single conservation law, indicating after the straightforward generalization to the vector case $m \geq 2$.

2.1. Reduction to constant coefficients.

Following [WZ3], we perturb $(A, B) = (A_0 e^{i(\kappa x - \omega t)}, B_0)$ as $A \to (A_0 + u + iv)e^{i(\kappa x - \omega t)}$ and $B \to B_0 + w$, with u, v, w real valued, giving linearized equations (1.12) in constant coefficient form. Here, we have used the identity

$$-i\omega U = -a\kappa^2 U + bU + cA_0^2 U + dB_0 U \tag{2.1}$$

for $U := u + iv$, coming from satisfaction of (mcGL) by the periodic solution.

To study stability, we compute the dispersion relations associated with (1.12). Namely, writing $(u, v, w) = (u_0, v_0, w_0)e^{i\hat\sigma x - \lambda t}$ we arrive at the eigenvalue problem

$$\lambda(u_0, v_0, w_0)^T = M(\varepsilon, \hat\sigma)(u_0, v_0, w_0)^T, \tag{2.2}$$

where

$$\begin{aligned} M(\varepsilon, \hat\sigma) = {} & \begin{pmatrix} 2A_0^2\Re(c) & 0 & A_0\Re(d) \\ 2A_0^2\Im(c) & 0 & A_0\Im(d) \\ 0 & 0 & 0 \end{pmatrix} + i\hat\sigma \begin{pmatrix} -2\kappa\Im(a) & -2\kappa\Re(a) & 0 \\ +2\kappa\Re(a) & -2\kappa\Im(a) & 0 \\ 2A_0 h\varepsilon^{-1} + 2A_0\kappa\Im(g) & 0 & \varepsilon^{-1} f \end{pmatrix} \\ & - \hat\sigma^2 \begin{pmatrix} \Re(a) & -\Im(a) & 0 \\ \Im(a) & \Re(a) & 0 \\ 2A_0\Re(g) & 2A_0\Im(g) & e_B \end{pmatrix} =: M_0 + \hat\sigma M_1 + \hat\sigma^2 M_2. \end{aligned} \tag{2.3}$$

Spectral stability, necessary for linearized stability, corresponds to

$$\Re\lambda \leq 0 \text{ for } \lambda = \lambda(\hat\sigma) \in \operatorname{spec}(M(\varepsilon, \hat\sigma)). \tag{2.4}$$

Diffusive spectral stability in the sense of Schneider [S1, S2], sufficient for linearized and nonlinear stability, corresponds to

$$\Re\lambda \leq c(\varepsilon)|\hat\sigma|^2/(1 + |\hat\sigma|^2) \text{ for } \lambda = \lambda(\hat\sigma) \in \operatorname{spec}(M(\varepsilon, \hat\sigma)). \tag{2.5}$$

Remark 2.1. We note the remarkable fact that B_0 does not appear in the linearized equations (1.12), having been removed using (2.1). Paradoxically, it thus appears not to enter stability considerations, despite its role through (1.11) in the existence theory. The resolution of this paradox is that the restriction (1.11) on existence stems from our convention (1.7), which will be seen to be necessary for stability. But apart from this indirect connection, indeed B_0 disappears in stability computations.

2.2. **Taylor expansion of the dispersion relations.** Our first task, carried out in the remainder of this section, is to compute an analytic expansion for the neutral dispersive curves about the origin, thereby determining *necessary conditions* for spectral and diffusive spectral stability. Our analysis follows to a large extent the corresponding analysis in [WZ3], in which a Taylor expansion was shown to exist but not explicitly computed. Here, we go substantially further, however, computing the Taylor coefficients to sufficiently high order in ε to obtain explicit low-frequency stability conditions.

To minimize the number of fractions in the following computation, define two parameters

$$p := -\frac{\Re(d)}{2A_0\Re(c)} \qquad q := -\frac{\Im(c)}{\Re(c)}. \tag{2.6}$$

so that

$$pq := \frac{\Re(d)\Im(c)}{2A_0\Re(c)^2}. \tag{2.7}$$

Remark 2.2. We recall from [WZ3] and (1.6) that p can be identified as

$$p = \frac{\partial A_0}{\partial B_0},$$

and that q is normalized measure of the nonlinear dispersion in the A-equation. Hence, in the vectorial case p will become a row vector and q will remain a scalar.

2.2.1. *Preliminary diagonalization.* At $\sigma = 0$, we have the (generalized) eigenvectors of $M(\varepsilon, 0)$ given by

$$R_s := \begin{pmatrix} 1 \\ -q \\ 0 \end{pmatrix}, \qquad R_t := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad R_c := \begin{pmatrix} p \\ 0 \\ 1 \end{pmatrix}. \tag{2.8}$$

In addition, we have corresponding left (generalized) eigenvectors

$$L_s := \begin{pmatrix} 1 & 0 & -p \end{pmatrix}, \qquad L_t := \begin{pmatrix} q & 1 & -pq \end{pmatrix}, \qquad L_c := \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \tag{2.9}$$

where (L_s, R_s) are the left/right eigenvector pair for the unique stable eigenvalue and (L_t, R_t), (L_c, R_c) are associated to the zero eigenvalue. We note that L_t and R_c are generalized left and right eigenvectors respectively, provided that $\Re(c)\Im(d) - \Im(c)\Re(d) \neq 0$.

Following [WZ3], this gives an initial block-diagonalization

$$\tilde{M}_0(\varepsilon) := \begin{pmatrix} L_s \\ L_t \\ L_c \end{pmatrix} M_0(\varepsilon) \begin{pmatrix} R_s & R_t & R_c \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & \hat{M}_0(\varepsilon) \end{pmatrix}, \tag{2.10}$$

where

$$m_0 := 2A_0^2\Re(c), \qquad \hat{M}_0 := \begin{pmatrix} 0 & A_0(\Im(d) + q\Re(d)) \\ 0 & 0 \end{pmatrix}. \tag{2.11}$$

Similarly, we find

(2.12)

$$\begin{aligned}
\tilde{M}_1(\varepsilon) &= i\varepsilon^{-1} \begin{pmatrix} -2A_0ph & 0 & -p(f + 2A_0hp) \\ -2A_0pqh & 0 & -pq(f + 2A_0hp) \\ 2A_0h & 0 & f + 2A_0hp \end{pmatrix} \\
&\quad + 2i\kappa \begin{pmatrix} -\Im(a) - A_0p\Im(g) + q\Re(a) & -\Re(a) & -p(\Im(a) + A_0p\Im(g)) \\ \Re(a) - A_0pq\Im(g) + q^2\Re(a) & -q\Re(a) - \Im(a) & p(-q\Im(a) + \Re(a) - A_0pq\Im(g)) \\ A_0\Im(g) & 0 & A_0\Im(g)p \end{pmatrix} \\
&= \begin{pmatrix} m_1(\varepsilon) & s_1(\varepsilon) \\ s_2(\varepsilon) & \hat{M}_1(\varepsilon) \end{pmatrix},
\end{aligned}$$

where

$$m_1(\varepsilon) = i\big(- 2\kappa\Im(a) - 2A_0\kappa p\Im(g) - 2\varepsilon^{-1}A_0ph + 2\kappa\Re(a)q\big)$$
$$= -2i\varepsilon^{-1}A_0ph + H.O.sT.\,, \tag{2.13}$$

(2.14)
$$\hat M_1(\varepsilon) = i\begin{pmatrix} -2\kappa(\Re(a)q + \Im(a)) & -pq\varepsilon^{-1}(f + 2A_0hp) + 2p\kappa\big(- \Im(a)q + \Re(a) - A_0pq\Im(g)\big) \\ 0 & \varepsilon^{-1}(f + 2A_0hp) + 2A_0\kappa\Im(g)p \end{pmatrix}$$
$$= i\begin{pmatrix} -2\kappa(\Re(a)q + \Im(a)) & -pq\varepsilon^{-1}(f + 2A_0hp) \\ 0 & \varepsilon^{-1}(f + 2A_0hp) \end{pmatrix} + H.O.T.\,,$$

$$s_1(\varepsilon) = i\begin{pmatrix}-2\kappa\Re(a) & -2\kappa\Im(a)p - 2\kappa pA_0\Im(g)p - \varepsilon^{-1}p(f + 2A_0hp)\end{pmatrix}$$
$$= -i\varepsilon^{-1}\begin{pmatrix}0 & p(f + 2A_0hp)\end{pmatrix} + i\begin{pmatrix}-2\kappa\Re(a) & O(1)\end{pmatrix}\,, \tag{2.15}$$

and

$$s_2(\varepsilon) = i\begin{pmatrix} -2\varepsilon^{-1}A_0pqh + 2\kappa\big(\Re(a) - A_0pq\Im(g) + q^2\Re(a)\big) \\ \varepsilon^{-1}2A_0h + 2A_0\kappa\Im(g)) \end{pmatrix}$$
$$= 2A_0i\varepsilon^{-1}\begin{pmatrix}-pqh \\ h\end{pmatrix} + i\begin{pmatrix}O(1) \\ 2A_0\kappa\Im(g))\end{pmatrix}\,. \tag{2.16}$$

Remark 2.3. In anticipation of the vectorial case, $m \geq 2$, we've carefully placed the h's, p's and $\Im(g)$'s, so that the corresponding expressions make sense when p is an $1 \times m$ row vector, f is an $m \times m$ matrix, and h, g are $m \times 1$ column vectors.

At next order, we obtain, likewise,

$$\tilde M_2(\varepsilon) = -\begin{pmatrix} m_2 & X_1 \\ X_2 & \hat M_2 \end{pmatrix}, \tag{2.17}$$

where

$$m_2 = \big(Re(a) + \Im(a)q - 2A_0p(\Re(g) - \Im(g)q)\big)\,,$$
$$X_1 = \big(\Im(a) - 2A_0p\Im(g) \quad p\Re(a) - 2A_0p\Re(g)p - pe_B\big)\,,$$
$$X_2 = \begin{pmatrix} \Im(a) - q^2\Im(a) - 2A_0pq\big(\Re(g) - \Im(g)q\big) \\ 2A_0\big(\Re(g) - \Im(g)q\big) \end{pmatrix}, \tag{2.18}$$
$$\hat M_2 = \begin{pmatrix} -q\Im(a) + \Re(a) - 2A_0pq\Im(g) & \big(q\Re(a) + \Im(a) - 2A_0pq\Im(g)\big)p - pqe_B \\ 2A_0\Im(g) & 2A_0\Re(g)p + e_B \end{pmatrix}.$$

2.2.2. *Some organizing discussion.* As described in [Wh, WZ3], the eigenvector R_t corresponds to a "translational mode" coming from rotational invariance in the complex Ginzburg-Landau equation for A, a consequence of translational invariance in the underlying PDE, while the eigenvector R_c corresponds to a "conservative mode" associated with the conservation law for B. The mode R_s corresponds to a (weakly) "stable mode" coming from supercritical Hopf bifurcation, as is familiar from the classical complex Ginzburg-Landau equation arising in the case without conservation laws.

These classifications may be helpful both in following the analysis, and in connecting to other work. For example, the neutral $\hat N$ block may be seen to correspond with the more general Whitham modulation approximation about arbitrary-amplitude waves, which, as described in [JNRZ], is precisely an expansion in neutral translational and conservative modes. As noted in [JZ1, JZ2, BJZ2] in the context of the Whitham equations, one finds from study of the corresponding existence problem that the lowest-order part $\hat N(0)$ generically consists of a nontrivial Jordan block, as we see also here, a consequence of the conservation structure of the equations.

Here as there, this leads to much of the difficulty in the study of stability; indeed, it is counterintuitive at first glance that spectra should expand analytically in $\hat\sigma$ instead of in a Puiseaux expansion in $\sqrt{\hat\sigma}$, as we shall see. The reason for this, as pointed out in [JZ2], is that the lower lefthand entry of the next-order block $\hat N_1$ opposite to the nonvanishing entry in the Jordan block necessarily vanishes, also by conservation principles, so that the matrix perturbation problem can be converted by a "balancing" transformation to a standard matrix perturbation problem without a nontrivial Jordan block. The latter procedure is described in Section 2.2.4 below.

Alternatively, looking at the exact spectral expansion of neutral modes for the underlying PDE problem, corresponding to variations along the manifold of periodic traveling-wave solutions, one finds [JZ2] that there is a Jordan block whenever speed of profiles is nonstationary, and that the corresponding generalized right zero eigenfunction lies in a Jordan chain over the genuine eigenfunction consisting of the spatial derivative $\partial_x\bar u$ of the background traveling wave. Meanwhile, associated left eigenfunctions are constant functions $\ell \equiv$ constant, by conservation form; the lower lefthand corner of the Jordan block, corresponding to $\langle \ell, \partial_x \bar u\rangle = \ell\cdot \bar(x)|_0^X$, X the period of the background wave, thus vanishes by periodicity of $\bar u$.

That is, *the same conservation structure leading to appearance of a Jordan block enforces constraints on the perturbation leading to analytic expansion nonetheless*: i.e., "the disease is also the cure." The importance of analytic vs. Puiseaux expansion is that the latter involves ill-conditioning of the spectral expansion/diagonalization procedure in the form of blowup of associated eigenprojectors. In the context of [JZ2], this would have wrecked the linearized stability estimates, leading to transient time-algebraic growth instead of decay. Here, it would have increased sensitivity to higher-order errors, preventing our estimates from closing.

2.2.3. *First-order diagonalization.* Following [MZ1, MZ], we apply the method of "successive diagonalization", defining a coordinate change

$$
\begin{aligned}
\mathcal{T} &= \begin{pmatrix} 1 & 0 \\ \hat\sigma t_2 & \mathrm{Id}_2 \end{pmatrix}\begin{pmatrix} 1 & \hat\sigma t_1 \\ 0 & \mathrm{Id}_2 \end{pmatrix} = \begin{pmatrix} 1 & \hat\sigma t_1 \\ \hat\sigma t_2 & \mathrm{Id}_2 + \hat\sigma^2 t_2 t_1 \end{pmatrix},\\
\mathcal{T}^{-1} &= \begin{pmatrix} 1 & -\hat\sigma t_1 \\ 0 & \mathrm{Id}_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\hat\sigma t_2 & \mathrm{Id}_2 \end{pmatrix} = \begin{pmatrix} 1+\hat\sigma^2 t_1 t_2 & -\hat\sigma t_1 \\ -\hat\sigma t_2 & \mathrm{Id}_2 \end{pmatrix},
\end{aligned}
\tag{2.19}
$$

with $t_1\hat M_0 - m_0 t_1 = -s_1$, $\hat M_0 t_2 - t_2 m_0 = s_2$, or equivalently

$$
\begin{aligned}
t_1 &= -s_1(\hat M_0 - m_0)^{-1},\\
t_2 &= (\hat M_0 - m_0)^{-1} s_2,
\end{aligned}
\tag{2.20}
$$

$\hat M_0$, m_0 as in (2.10), in order to diagonalize to order $O(\hat\sigma)$. With this choice, we obtain

$$
N(\hat\sigma,\varepsilon) := \mathcal{T}\tilde M\mathcal{T}^{-1} = N_0(\varepsilon) + \hat\sigma N_1(\varepsilon) + \hat\sigma^2 N_2(\varepsilon) + \hat\sigma^3 N_3(\hat\sigma) + O(\hat\sigma^4),
\tag{2.21}
$$

where

$$
\begin{aligned}
N_0 = \tilde M_0 &= \begin{pmatrix} m_0 & 0 \\ 0 & \hat M_0 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} m_1 & 0 \\ 0 & \hat M_1 \end{pmatrix},\\
N_2 &= \begin{pmatrix} * & * \\ * & \hat N_2 \end{pmatrix}, \qquad N_3 = \begin{pmatrix} * & * \\ * & \hat N_3 \end{pmatrix},
\end{aligned}
\tag{2.22}
$$

with

$$\begin{aligned}\hat N_2 &= -\hat M_2 + (t_2 s_1 - s_2 t_1) + t_2 t_1 \hat M_0 - t_2 m_0 t_1\\ &= -\hat M_2 + (t_2 s_1 - s_2 t_1) - t_2 s_1\\ &= -\hat M_2 - s_2 t_1\\ &= -\hat M_2 + s_2 s_1 (\hat M_0 - m_0)^{-1}\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}\hat N_3 &= -t_2 m_1 t_1 + t_2 t_1 \hat M_1 - t_2 X_1 + X_2 t_1 = t_2 t_1 (\hat M_1 - m_1) - t_2 X_1 + X_2 t_1\\ &= -(\hat M_0 - m_0)^{-1} s_2 s_1 (\hat M_0 - m_0)^{-1} (\hat M_1 - m_1) - t_2 X_1 + X_2 t_1.\end{aligned} \tag{2.24}$$

Here, the recurring term $s_2 s_1$ by direct computation is

$$s_2 s_1 = 2\varepsilon^{-2} A_0 \begin{pmatrix} 0 & -qphp(f + 2A_0 hp)\\ 0 & hp(f + 2A_0 hp)\end{pmatrix} + \varepsilon^{-1} \begin{pmatrix} -4A_0 \kappa \Re(a) pqh & O(1)\\ 4A_0 \kappa h \Re(a) & O(1)\end{pmatrix}. \tag{2.25}$$

Applying standard spectral perturbation theory [K], we have that there exists an *exact* analytic in $\hat\sigma$ change of coordinates decoupling n and $\hat N$ blocks for which the above approximate series are correct to $O(\hat\sigma^4)$ error. We thus have that the "stable" eigenvalue λ_s corresponding to the n block satisfies

$$\lambda_s(\varepsilon) = m_0 + O(\hat\sigma) = 2A_0^2 \Re(c) + O(\hat\sigma). \tag{2.26}$$

The "translational" and "conservative" eigenvalues λ_t and λ_c can then be determined by analysis of the reduced matrix perturbation problem

$$\hat N(\varepsilon, \hat\sigma) := \hat N_0(\varepsilon) + \hat\sigma \hat N_1(\varepsilon) + \hat\sigma^2 \hat N_2(\varepsilon) + \hat\sigma^3 \hat N_3(\varepsilon). \tag{2.27}$$

Note, by supercriticality condition (1.7), that $m_0 = 2A_0^2\Re(c) < 0$, so that indeed the "stable" eigenvalue is stable. This observation indirectly justifies our convention in assuming (1.7) in the existence problem, since otherwise the resulting solutions would be *exponentially unstable* by (2.26).

Remark 2.4. The analogous computation in [WZ3] uses a more traditional spectral perturbation argument. In order to connect this argument to that argument, we note that the first order diagonalization can also be interpreted as finding the first correctors of the left/right eigenvectors (L_s, R_s) associated to the stable eigenvalue $\lambda_s(0) = m_0$, as we recall from [K] that simple eigenvalues have smooth eigenvectors.

2.2.4. *Balancing transformation.* In the generic case

$$\Im(d) \neq \Re(d)\Im(c)/\Re(c), \tag{2.28}$$

$\hat M_0$ takes the form of a (nonzero multiple of a) Jordan block

$$\hat M_0 = rJ; \qquad J = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}, \quad r := A_0(\Im(d) - \Re(d)\Im(c)/\Re(c)) \neq 0, \tag{2.29}$$

giving

$$(\hat M_0 - m_0)^{-1} = \big(-m_0(I - (r/m_0)J)\big)^{-1} = -m_0^{-1}\big(I + (r/m_0)J\big) \tag{2.30}$$

and, by (2.25)

$$\begin{aligned} s_2 s_1 (\hat M_0 - m_0)^{-1} &= -s_2 s_1 m_0^{-1}\big(I + (r/m_0)J\big).\\ &= -\frac{1}{m_0} s_2 \begin{pmatrix} -2i\kappa\Re(a) & -i\varepsilon^{-1} p(f + 2A_0 hp) + O(1)\end{pmatrix} \begin{pmatrix} 1 & \frac{r}{m_0}\\ 0 & 1\end{pmatrix}\\ &= -\frac{1}{m_0} s_2 s_1 + \begin{pmatrix} 0 & O(\varepsilon^{-1})\\ 0 & O(\varepsilon^{-1})\end{pmatrix},\end{aligned} \tag{2.31}$$

$$(\hat M_1 - m_1) = i\varepsilon^{-1}\begin{pmatrix} 2A_0ph & -pq(f+2A_0hp) \\ 0 & f+2A_0hp+2A_0ph \end{pmatrix} + O(1), \tag{2.32}$$

and

$$\begin{aligned}(\hat M_0 - m_0)^{-1}s_2s_1(\hat M_0 - m_0)^{-1}(\hat M_1 - m_1) &= i\frac{\varepsilon^{-2}}{m_0^2}\begin{pmatrix} 1 & r/m_0 \\ 0 & 1\end{pmatrix} \\ &\times\begin{pmatrix} -4A_0\kappa\Re(a)pqh & -2A_0qphp(f+2A_0hp)\varepsilon^{-1} \\ 4A_0\kappa h\Re(a) & 2A_0hp(f+2A_0hp)\varepsilon^{-1}\end{pmatrix} \\ &\times\begin{pmatrix} 1 & r/m_0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 2A_0ph & -pq(f+2A_0hp) \\ 0 & f+2A_0hp+2A_0ph\end{pmatrix} + O(\varepsilon^2) \\ &= i\frac{\varepsilon^{-2}}{m_0^2}\begin{pmatrix} O(\varepsilon^{-1}) & O(\varepsilon^{-1}) \\ 8A_0^2\kappa\Re(a)hph & O(\varepsilon^{-1})\end{pmatrix} + H.O.T.\end{aligned} \tag{2.33}$$

Remark 2.5. The distinction between $2A_0hp$ and $2A_0ph$ in (2.32) is being kept in order to more easily adapt this calculation to the systems case.

We note here that the crucial fact that makes this entire calculation of dispersion relations possible is that, while $\hat N_3$ is of order ε^{-3}, $\hat N_3$ is upper triangular to leading order. Hence the bottom left corner of $\hat N_3$ is of order ε^{-2}, matching the order of $\hat N_2$. The importance of this observation comes from the "balancing" transformation which informally takes the top right corner and moves it up an order and similarly takes the bottom left corner and moves it down an order. We also note here that (2.32) is upper triangular, and that the $O(1)$ terms are proportional to κ.

Thus,

$$\hat N_1 = \hat M_1 = i\begin{pmatrix} -2\kappa(\Re(a)q + \Im(a)) & -pq\varepsilon^{-1}(f+2A_0hp) \\ 0 & \varepsilon^{-1}(f+2A_0hp)\end{pmatrix} + H.O.T.\,, \tag{2.34}$$

$$\begin{aligned}\hat N_2 &= -\hat M_2 + s_2s_1(\hat M_0 - m_0)^{-1} \\ &= -m_0^{-1}\varepsilon^{-1}\begin{pmatrix} -4A_0\kappa\Re(a)pqh & -2A_0qphp(f+2A_0hp)\varepsilon^{-1} \\ 4A_0\kappa h\Re(a) & 2A_0hp(f+2A_0hp)\varepsilon^{-1}\end{pmatrix} + H.O.T.\ ,\end{aligned} \tag{2.35}$$

and

$$\hat N_3 = -(\hat M_0 - m_0)^{-1}s_2s_1(\hat M_0 - m_0)^{-1}(\hat M_1 - m_1) - t_2X_1 + X_2t_1 = \begin{pmatrix} * & * \\ O(\varepsilon^{-2}) & *\end{pmatrix}. \tag{2.36}$$

Remark 2.6. Evidently, for the decoupled case $d = 0$ considered in [HSZ], (2.28) fails, hence that case is degenerate from this point of view. The case that (2.28) fails but $d \neq 0$ is treated in [WZ3].

To remove the Jordan block, following [MZ1, MZ], we perform the "balancing" transformation $\hat N \to O := \mathcal{S}\hat N\mathcal{S}^{-1}$, where

$$\mathcal{S} := \begin{pmatrix} i\hat\sigma & 0 \\ 0 & 1\end{pmatrix}, \quad \mathcal{S}^{-1} = \begin{pmatrix} (i\hat\sigma)^{-1} & 0 \\ 0 & 1\end{pmatrix}, \tag{2.37}$$

yielding

$$O(\varepsilon,\hat\sigma) = \hat\sigma O_1 + \hat\sigma^2 O_2 + O(\hat\sigma^3), \tag{2.38}$$

with

$$
\begin{aligned}
O_1 &= i\begin{pmatrix} -2\kappa(\Re(a)q+\Im(a)) & r \\ 4\kappa\varepsilon^{-1}m_0^{-1}A_0h\Re(a) & \varepsilon^{-1}(f+2A_0hp)\end{pmatrix} + H.O.T. \\
&= i\begin{pmatrix} 0 & 0 \\ 4\kappa m_0^{-1}\varepsilon^{-1}A_0h\Re(a) & \varepsilon^{-1}(f+2A_0hp)\end{pmatrix} + H.O.T., \\
O_2 &= \begin{pmatrix} 4\varepsilon^{-1}m_0^{-1}A_0\kappa\Re(a)pqh & pq\varepsilon^{-1}(f+2A_0hp) \\ -8\varepsilon^{-2}m_0^{-2}A_0^2\kappa(hph\Re(a)) & -\varepsilon^{-2}m_0^{-1}2A_0hp(f+2A_0hp)\end{pmatrix} + H.O.T.
\end{aligned}
\tag{2.39}
$$

This can now be expanded up to order $\hat\sigma^2$ by standard perturbation of distinct eigenvalues, as the eigenvalues of O_1 are $\sim 1, \varepsilon^{-1}$ and split for $\varepsilon > 0$ sufficiently small under the generic condition

$$
f + 2A_0hp \neq 0. \tag{2.40}
$$

Moreover (by splitting), we may conclude *analyticity in* $\hat\sigma$ on some sufficiently small ball.

Namely, taking the eigenvectors to lowest order in ε of O_1, we have

$$
\begin{aligned}
&\ell_t = \begin{pmatrix}1 & 0\end{pmatrix} + O(\varepsilon), \quad \ell_c = \begin{pmatrix}4\kappa m_0^{-1}\Re(a)A_0(f+2A_0hp)^{-1}h & 1\end{pmatrix} + O(\varepsilon) \\
&r_t = \begin{pmatrix} 1 \\ -4\kappa m_0^{-1}A_0\Re(a)(f+2A_0hp)^{-1}h\end{pmatrix} + O(\varepsilon), \quad r_c = \begin{pmatrix}0\\1\end{pmatrix} + O(\varepsilon),
\end{aligned}
\tag{2.41}
$$

giving the analytic expansion

$$
\begin{aligned}
\lambda_t &= \alpha_t\hat\sigma + \mu_t\hat\sigma^2 + O(\hat\sigma^3), \\
\lambda_c &= \alpha_c\hat\sigma + \mu_c\hat\sigma^2 + O(\hat\sigma^3),
\end{aligned}
\tag{2.42}
$$

where α_j are pure imaginary

$$
\begin{aligned}
\alpha_t &= \ell_tO_1r_t = O(1), \\
\alpha_c &= \ell_cO_1r_c = i\varepsilon^{-1}(f+2A_0hp) + O(1),
\end{aligned}
\tag{2.43}
$$

are *pure imaginary* and

$$
\begin{aligned}
\mu_t &= \ell_tO_2r_t = O(\varepsilon^{-1}) =: \varepsilon^{-1}\mu_t^0 + O(1), \\
\mu_c &= \ell_cO_2r_c = -2\varepsilon^{-2}m_0^{-1}A_0hp(f+2A_0hp) + O(\varepsilon^{-1}) =: \varepsilon^{-2}\mu_c^0 + H.O.T..
\end{aligned}
\tag{2.44}
$$

Thus, *the signs of the real parts of* λ_t *and* λ_c *are determined by the signs of* $\Re\mu_t$ *and* $\Re\mu_c$.

Remark 2.7. In the vectorial case $m > 1$, the lower righthand block of O_1 is an $m \times m$ matrix with leading order $\varepsilon^{-1}(f + 2A_0hp)i$, hence requires the additional, first-order stability condition (1.23) in order to ensure that its spectra are pure imaginary. The second order stability conditions also require modification, see Section 5 for details.

Remark 2.8. In the systems setting, $\ell_c \to \ell_{c,i}$ with first entry corresponding to the i-th entry of the vector $4\kappa m_0^{-1}\Re(a)A_0(f + 2A_0hp)^{-1}h$ and 1 replaced with the i-th standard basis of $\mathbb{R}^m$, thought of as a row vector. Similarly, $r_c \to r_{c,i}$ with 1 replaced with the i-th standard basis element of $\mathbb{R}^m$, now thought of as a column vector. Finally, we note that to complete the eigenvectors we have that $\ell_t \to \begin{pmatrix}1 & 0\end{pmatrix}$ and $r_t \to \begin{pmatrix} 1 \\ -4\kappa m_0^{-1}\Re(a)A_0(f+2A_0hp)^{-1}h\end{pmatrix}$.

However, our expansion is not fine enough to determine μ_t, or even its order, i.e., whether μ_t^0 is vanishing or nonvanishing. This could be remedied by applying the method of repeated diagonalization in powers of ε to expand (2.41) to as many orders as required to obtain a nonvanishing coefficient in the expansion of μ_t. In [WZ3], μ_t was computed through a Kato-style expansion when

$r = 0$, i.e. $\hat{M}_0$ is not a Jordan block. In that case, $\mu_t^0 = 0$, or equivalently, $\mu_t(\varepsilon) = O(1)$.

While a similar expansion is possible here, it is significantly more complicated due to α_t and μ_t being primarily determined by error terms in the O_j. Instead, we elect to work with (2.3) directly and use the method of matched asymptotics. It is at this point that our argument specializes to the case of a single conservation law, see Section 5.1 for a variation on this argument for the case $m \geq 2$. Suppose that (2.3) has an eigenvalue of the form $\lambda_t(\hat{\sigma}) = i\alpha_t\hat{\sigma} + \mu_t\hat{\sigma}^2$ with $\alpha_t = O(1)$ and $\mu_t = O(\varepsilon^{-1})$. Recalling $M(\varepsilon, \hat{\sigma})$ from (2.3), we find

$$M(\varepsilon,\hat{\sigma}) = -\hat{\sigma}^2 \begin{pmatrix} \Re(a) & -\Im(a) & 0 \\ \Im(a) & \Re(a) & 0 \\ 2A_0\Re(g) & 2A_0\Im(g) & e_B \end{pmatrix} + i\hat{\sigma} \begin{pmatrix} -2\kappa\Im(a) & -2\kappa\Re(a) & 0 \\ +2\kappa\Re(a) & -2\kappa\Im(a) & 0 \\ 2A_0h\varepsilon^{-1} + 2A_0\kappa\Im(g) & 0 & \varepsilon^{-1}f \end{pmatrix}$$
$$+ \begin{pmatrix} 2A_0^2\Re(c) & 0 & A_0\Re(d) \\ 2A_0^2\Im(c) & 0 & A_0\Im(d) \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us make the important observation that because $\lambda_t = O(\hat{\sigma})$, the second column and third row of $M(\varepsilon, \hat{\sigma}) - \lambda(\hat{\sigma})Id$ are both proportional to $i\hat{\sigma}$, with the intersection being order $\hat{\sigma}^2$, and so can be divided out. Hence, to determine α_t, we need to work with the now $O(1)$ terms in the characteristic polynomial given by

$$P_0 := \det \begin{pmatrix} 2A_0^2\Re(c) & -2\kappa\Re(a) & A_0\Re(d) \\ 2A_0^2\Im(c) & -2\kappa\Im(a) - \alpha_t & A_0\Im(d) \\ 2A_0h\varepsilon^{-1} + 2A_0\kappa\Im(g) & 2A_0\Im(g) & \varepsilon^{-1}f - \alpha_t \end{pmatrix}. \tag{2.45}$$

Expanding P_0 along the first column gives us

$$\begin{aligned} P_0 &= 2A_0^2\Re(c)\alpha_t^2 + \big(\varepsilon^{-1}2A_0^2(h\Re(d) - f\Re(c)) + O(1)\big)\alpha_t \\ &\quad + \varepsilon^{-1}4A_0^2\kappa\big(f\Re(a)\Im(c) - f\Im(a)\Re(c) + h\Im(a)\Re(d) - h\Re(a)\Im(d)\big) + O(1) \\ &=: c_2\alpha_t^2 + c_1\alpha_t + c_0. \end{aligned} \tag{2.46}$$

Now, if λ_t is to be an eigenvalue, we need $P_0 = 0$. Applying the quadratic formula to (2.46), we find that the two roots of P_0 are

$$\alpha = \frac{-c_1 \pm \sqrt{c_1^2 - 4c_2c_0}}{2c_2}.$$

As $c_2 = O(1)$ and $c_1, c_0 = O(\varepsilon^{-1})$, we see that the c_1^2 term in the square root dominates. Hence applying the binomial theorem we find the two roots are

$$\alpha_- = -\frac{c_1}{c_2} + O(1), \qquad \alpha_+ = -\frac{c_0}{c_1} + O(\varepsilon).$$

Recalling the expressions in (2.46), we find that

$$\alpha_- = \varepsilon^{-1}\Big(f - \frac{h\Re(d)}{\Re(c)}\Big) + O(1).$$

Upon recalling the definition of p from (2.6), we notice that $\alpha_- = \alpha_c$ for α_c as in (2.43). Evidently, we then conclude that $\alpha_+ = \alpha_t$. Returning to (2.46), we find that

$$\alpha_t = -2\kappa\frac{f\Re(a)\Im(c) - f\Im(a)\Re(c) + h\Im(a)\Re(d) - h\Re(a)\Im(d)}{h\Re(d) - f\Re(c)} + O(\varepsilon). \tag{2.47}$$

On the other hand, using (2.41), (2.39), and a Kato-style expansion, we can also find

$$\alpha_t = -2\kappa(\Re(a)q + \Im(a)) - \frac{4\kappa A_0\Re(a)}{m_0(f + 2A_0hp)}rh + O(\varepsilon). \tag{2.48}$$

To get from (2.47) to (2.48), we start by observing that

$$h\Re(d) - f\Re(c) = -\Re(c)\big(f + 2A_0hp\big).$$

We then see that the two central terms in the numerator of (2.47) are $\Im(a)$ times the denominator and so we obtain

$$\alpha_t = -2\kappa\Big(\Im(a) - \Re(a)\frac{f\Im(c) - h\Im(d)}{\Re(c)(f + 2A_0hp)}\Big) + O(\varepsilon).$$

In the numerator, we now add and subtract $2A_0hp\Im(c)$ and cancel with the denominator to obtain

$$\alpha_t = -2\kappa\Big(\Im(a) - \Re(a)\frac{\Im(c)}{\Re(c)} + h\Re(a)\frac{\Im(c)2A_0p + \Im(d)}{\Re(c)(f + 2A_0hp)}\Big) + O(\varepsilon).$$

Now, plugging in the value of p and recalling the definition of r, we come to

$$\alpha_t = -2\kappa\Big(\Im(a) - \Re(a)\frac{\Im(c)}{\Re(c)} + h\Re(a)\frac{r}{2A_0\Re(c)(f + 2A_0hp)}\Big) + O(\varepsilon).$$

We note that upon plugging in $m_0 = 2A_0^2\Re(c)$ that this expression matches (2.48).

Turning to μ_t, we start by finding the coefficient, P_1, of $i\hat{\sigma}$ in $(i\hat{\sigma})^{-2}\det(M(\varepsilon,\hat{\sigma}) - \lambda_t Id)$. Using multilinearity of det, we find by choosing the $O(\hat{\sigma})$ terms in each column and the $O(1)$ terms in the remaining columns

(2.49)

$$P_1 = \det\begin{pmatrix} -2\kappa\Im(a) - \alpha_t & -2\kappa\Re(a) & A_0\Re(d) \\ 2\kappa\Re(a) & -2\kappa\Im(a) - \alpha_t & A_0\Im(d) \\ 2A_0\Re(g) & 2A_0\Im(g) & \varepsilon^{-1}f - \alpha_t \end{pmatrix} + \det\begin{pmatrix} 2A_0^2\Re(c) & -\Im(a) & A_0\Re(d) \\ 2A_0^2\Im(c) & \Re(a) + \mu_t & A_0\Im(d) \\ 2A_0h\varepsilon^{-1} + 2A_0\kappa\Im(g) & 0 & \varepsilon^{-1}f - \alpha_t \end{pmatrix}$$
$$+ \det\begin{pmatrix} 2A_0^2\Re(c) & -2\kappa\Re(a) & 0 \\ 2A_0^2\Im(c) & -2\kappa\Im(a) & 0 \\ 2A_0h\varepsilon^{-1} + 2A_0\kappa\Im(g) & 2A_0\Im(g) & e_B + \mu_t \end{pmatrix}.$$

The first determinant in (2.49) is to leading order in ε given by

(2.50)

$$\det\begin{pmatrix} -2\kappa\Im(a) - \alpha_t & -2\kappa\Re(a) & A_0\Re(d) \\ 2\kappa\Re(a) & -2\kappa\Im(a) - \alpha_t & A_0\Im(d) \\ 2A_0\Re(g) & 2A_0\Im(g) & \varepsilon^{-1}f - \alpha_t \end{pmatrix} = \varepsilon^{-1}f\Big((-2\kappa\Im(a) - \alpha_t)^2 + 4\kappa^2\Re(a)^2\Big) + O(1).$$

For the second determinant in (2.49), we obtain

(2.51)

$$\det\begin{pmatrix} 2A_0^2\Re(c) & -\Im(a) & A_0\Re(d) \\ 2A_0^2\Im(c) & \Re(a) + \mu_t & A_0\Im(d) \\ 2A_0h\varepsilon^{-1} + 2A_0\kappa\Im(g) & 0 & \varepsilon^{-1}f - \alpha_t \end{pmatrix} = \mu_t\big(2A_0^2\varepsilon^{-1}(\Re(c)f - h\Re(d)) + O(1)\big)$$
$$+ 2\varepsilon^{-1}A_0^2\Im(a)\big(f\Im(c) - h\Im(d)\big)$$
$$+ 2\varepsilon^{-1}A_0^2\Re(a)\big(\Re(c)f - h\Re(d)\big) + O(1).$$

The third determinant is linear in μ_t and $O(1)$, so we will not need it. So, setting $P_1 = 0$, we obtain a linear equation of the form

$$c_3\mu_t + c_4 = 0,$$

with $c_3, c_4 = O(\varepsilon^{-1})$. Applying (2.50) and (2.51), we are led to

(2.52)

$$\mu_t = -\frac{f\Big((-2\kappa\Im(a) - \alpha_t)^2 + 4\kappa^2\Re(a)^2\Big) + 2A_0^2\Im(a)\big(f\Im(c) - h\Im(d)\big) + 2A_0^2\Re(a)\big(f\Re(c) - h\Re(d)\big)}{2A_0^2(\Re(c)f - h\Re(d))}$$
$$+ O(\varepsilon).$$

Remark 2.9. The same kind of calculation can also be used to find μ_c. Indeed, as $\alpha_c \sim \varepsilon^{-1}$, one replaces (2.50) with

$$(\varepsilon^{-1} f - \alpha_c)\alpha_c^2 + O(\varepsilon^{-2}).$$

We must also adapt (2.51) to

$$\mu_c 2A_0^2\Big(\Re(c)\big(\varepsilon^{-1} f - \alpha_c\big) - \varepsilon^{-1} h\Re(d)\Big) + H.O.T.$$

We notice that this expression vanishes to leading order. In order to match the $O(\varepsilon^{-3})$ term from (2.50), we consider the third matrix in (2.49), which has an $O(\varepsilon^{-1})$ coefficient of μ_c coming from the third determinant in (2.49), which gives us

$$-2A_0^2\Re(c)\alpha_c\mu_c + H.O.T.$$

At this point, we then obtain

$$\mu_c = \frac{(\varepsilon^{-1} f - \alpha_c)\alpha_c}{2A_0^2\Re(c)} + O(\varepsilon^{-1}).$$

Plugging in the expression for α_c, we obtain

$$\mu_c = \varepsilon^{-2}\frac{\Re(d)h}{2A_0^2\Re(c)^2}\Big(f - \frac{\Re(d)h}{\Re(c)}\Big) + O(\varepsilon^{-1}),$$

which upon factoring a copy $f/\Re(c)$ out of the expression in parentheses yields

$$\mu_c = \varepsilon^{-2}\frac{\Re(d)hf}{2A_0^2\Re(c)^3}\Big(\Re(c) - \frac{\Re(d)h}{f}\Big) + O(\varepsilon^{-1}),$$

which agrees with our previous expression.

Let us make a few key observations about (2.52). The first is that, to leading order, μ_t is a function of κ^2 as α_t in (2.47) is to leading order proportional to κ. Second, μ_t is, to leading order, *independent* of e_B and g. Third, at $\kappa = 0$, μ_t reduces to

$$\mu_t = -\Re(a) - \Im(a)\frac{\Im(\hat{c})}{\Re(\hat{c})}, \tag{2.53}$$

where $\hat{c} = c - dh/f$ is the updated value of c in the Darcy reduction. Hence the corresponding Benjamin-Feir-Newell criterion is the same as that for the Darcy reduction provide that $\Re(\hat{b}) > 0$.

Let us start by finding necessary stability criteria, starting with μ_c. We observe in the scalar case that the sign of μ_c^0 is *independent* of κ, as by (2.44), (2.6), (2.13), we see that

$$\mu_c^0 = -2m_0^{-1}A_0hp(f + 2A_0hp) = \frac{\Re(d)hf}{2A_0^2\Re(c)^3}\Big(\Re(c) - \frac{\Re(d)h}{f}\Big), \tag{2.54}$$

which exactly matches the corresponding expression in [WZ3]. Indeed, this leads us to half of the stability criterion for μ_c, extending Lemma 5.6 of [WZ3] whose proof we also recall.

Lemma 2.10. *If $\Re(\hat{c}) > 0$ with $\hat{c} = c - dh/f$, then $\mu_c > 0$.*

Proof. If $\Re(\hat{c}) > 0$, then $\Re(d)h/f < 0$ since $\Re(c) < 0$. Hence in the right most expression of (2.54), the term in parentheses is positive and the term outside parentheses is also positive since $\Re(d)h/f$ and $\Re(d)hf$ have the same sign and under the assumption $\Re(\hat{c}) > 0$ has the same sign as $\Re(c)^3$. □

Let us proceed further to find necessary and sufficient criteria for $\mu_c^0 < 0$.

Proposition 2.11. *There holds $\mu_c^0 < 0$ if and only if*

$$\Re(c) < \frac{\Re(d)h}{f} < 0. \tag{2.55}$$

Proof. The lower inequality in (2.55) was established in Lemma 2.10. For the other inequality, assume $\Re(\hat{c}) < 0$ and so $\mu_c^0 < 0$ if and only if $\Re(d)hf < 0$ as $\Re(\hat{c})/\Re(c)^3 > 0$ under the assumption $\Re(\hat{c}) < 0$. This completes the proof. □

To obtain Eckhaus-style criteria from (2.52), we begin by multiplying and dividing by f. After doing so and using $\hat{c} = c - dh/f$, we obtain

$$\mu_t = -\frac{\Big((-2\kappa\Im(a) - \alpha_t)^2 + 4\kappa^2\Re(a)^2\Big) + 2A_0^2\Im(a)\Im(\hat{c}) + 2A_0^2\Re(a)\Re(\hat{c})}{2A_0^2\Re(\hat{c})} + O(\varepsilon). \tag{2.56}$$

To determine the sign μ_t, we observe that we can freely drop the $2A_0^2$ from the denominator as it is always positive. By Lemma 2.10, we can further assume that $\Re(\hat{c}) < 0$ as otherwise all waves are unstable. So for stability, we're left with determining for what κ

$$\Big((-2\kappa\Im(a) - \alpha_t)^2 + 4\kappa^2\Re(a)^2\Big) + 2A_0^2\Im(a)\Im(\hat{c}) + 2A_0^2\Re(a)\Re(\hat{c}) < 0, \tag{2.57}$$

holds. To continue, we do the same to α_t as in (2.47), where we obtain

$$\alpha_t = 2\kappa\Big(- \Im(a) + \Re(a)\frac{\Im(\hat{c})}{\Re(\hat{c})}\Big).$$

Recalling the definition of A_0^2 from (1.6), we find that (2.57) can be written as

$$4\kappa^2\Re(a)^2(1+\hat{q}^2) - 2\frac{\Re(\tilde{b}) - \Re(a)\kappa^2}{\Re(c)}(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})) < 0, \tag{2.58}$$

where $\hat{q} := -\Im(\hat{c})/\Re(\hat{c})$. Collecting the κ^2 terms on the left hand side and the remainder on the right hand side, we find

$$\kappa^2\Big(4\Re(a)^2(1+\hat{q}^2) + 2\frac{\Re(a)}{\Re(c)}\big(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})\big)\Big) < 2\frac{\Re(\tilde{b})}{\Re(c)}(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})). \tag{2.59}$$

For us, the Benjamin-Feir-Newell criterion is that (2.53) is negative. Or, after rearranging,

$$\Re(a) + \Im(a)\frac{\Im(\hat{c})}{\Re(\hat{c})} > 0.$$

As we've assumed $\Re(\hat{c}) < 0$, we then obtain

$$\Re(a)\Re(\hat{c}) + \Im(a)\Im(\hat{c}) < 0.$$

Moreover, since we've assumed $\Re(c) < 0$, $\Re(a) > 0$, and $\Re(\tilde{b}) > 0$, we find that the coefficients in (2.59) are positive.

Lemma 2.12. *Let κ_S^2 be defined by*

$$\kappa_S^2 := \frac{2\frac{\Re(\tilde{b})}{\Re(c)}(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c}))}{4\Re(a)^2(1+\hat{q}^2) + 2\frac{\Re(a)}{\Re(c)}\big(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})\big)}. \tag{2.60}$$

Then $\kappa_S^2 \le \kappa_E^2$ for κ_E^2 as in (1.8) *provided the Benjamin-Feir-Newell criterion* (2.53) *holds.*

Proof. In (2.59), we observe that the left hand side is bounded from below by

$$\kappa^2\Big(2\frac{\Re(a)}{\Re(c)}\big(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})\big)\Big),$$

as $4\Re(a)^2(1+\hat{q}^2)\kappa^2$ is readily seen to be positive. As κ_S^2 is the right hand side of (2.59) divided by the left hand side, we conclude that

$$\kappa_S^2 < \frac{2\frac{\Re(\tilde{b})}{\Re(c)}\big(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})\big)}{2\frac{\Re(a)}{\Re(c)}\big(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})\big)} = \kappa_E^2.$$

□

Remark 2.13. One can expect that μ_t and μ_c are real based on an observation which played a key role in [WZ3] in showing that the wave could not destabilize to leading order. More precisely, each branch of the dispersion relation satisfies

$$\overline{\lambda(-\sigma)} = \lambda(\sigma), \tag{2.61}$$

near $\sigma = 0$. Hence by (2.61) and the chain rule, all even derivatives of λ are real at $\sigma = 0$ and all odd derivatives of λ are pure imaginary at $\sigma = 0$.

Definition 2.14. We define *asymptotic diffusive stability* by the pair of conditions

$$\Re\mu_t^0 > 0,\ \Re\mu_c^0 > 0, \text{ with } \mu_j \text{ as in (2.44).} \tag{2.62}$$

We define *asymptotic instability* by failure of *asymptotic neutral stability*

$$\Re\mu_t^0 \geq 0,\ \Re\mu_c^0 \geq 0, \tag{2.63}$$

i.e., $\Re\mu_t^0 < 0$ or $\Re\mu_c^0 < 0$.[2]

Evidently, asymptotic diffusive stability is necessary for strict stability, $\Re\lambda(\varepsilon,\delta) < 0$ for all $\varepsilon > 0$, $\delta \neq 0$, while asymptotic instability is sufficient for instability, $\Re\lambda(\varepsilon,\delta) > 0$ for some $\varepsilon > 0$, $\delta \in \mathbb{R}$.

Remark 2.15. In the decoupled case $d = 0$ treated in [HSZ], we see that $\mu_c = 0$ and diffusion is at the lower order e_B. That is, the ε^{-2} diffusion rate in the generic case is an example of "convection-enhanced diffusion".

Remark 2.16. The key quantity $f + 2A_0hp$ in (2.40) may be regarded as an effective convection obtained by Chapman-Enskog type reduction; note that our equations have the form of a relaxation system. The condition that $f + 2A_0hp$ be nonzero is equivalent to noncharacteristicity of this effective convection. Indeed, by the desingularizing rescaling $\delta \to \tilde{\delta}$, we are effectively converting the singular truncated model to a *singularly perturbed* model

$$\begin{aligned} A_t &= bA + c|A|^2A + dAB\varepsilon A_{xx}, \\ B_t + (f - h|A|^2)_x &= \varepsilon(\Re(gA\bar{A}_x)_x + \varepsilon e_B B_{xx} \end{aligned} \tag{2.64}$$

consisting of an inviscid limit of a relaxation system, similar in form to the viscous Saint Venant equations treated in [BJNRZ]. This may further explain points of similarity in the analyses remarked elsewhere. We note in particular the remarkably simple form of μ_c as proportional to the effective convection, similar to a key relation found in [JNRYZ], so that *second-order behavior in the "mean mode" c may be deduced by study of more easily accessible first-order behavior in* $\hat{\sigma}$, e.g., by the first-order Whitham modulation approximation.

Remark 2.17. The degenerate case $\Im(d) = q\Re(d)$ treated already in [WZ3, Wh] may be treated by the above argument equally well, that is, this covers both degenerate and generic case in a common framework. However, more terms must be included in deriving asymptotics.

[2]As μ_t^0 and μ_c^0 are real in this setting, these are sign conditions for μ_t^0 and μ_c^0 themselves.

3. Necessity of Darcy conditions

We next establish necessity of the Darcy stability conditions $\Re(\hat\sigma) < 0$, $\Re\mu_t^0 < 0$, determining stability of periodic solutions with respect to the reduced Darcy system (1.18), for stability with respect to the full system (mcGL).

3.1. **Relation of linearizations.** Our first step is to observe the relation between the linearizations of (mcGL) and (1.18) after reduction to constant coefficients. Recall that the linearization of the full system (mcGL), reduced to constant coefficients, is

(3.1)
$$M(\varepsilon,\sigma) = -\sigma^2 \begin{pmatrix} \Re(a) & -\Im(a) & 0 \\ \Im(a) & \Re(a) & 0 \\ 2A_0\Re(g) & 2A_0\Im(g) & e_B \end{pmatrix} + i\sigma \begin{pmatrix} -2\kappa\Im(a) & -2\kappa\Re(a) & 0 \\ +2\kappa\Re(a) & -2\kappa\Im(a) & 0 \\ 2A_0h\varepsilon^{-1} + 2A_0\kappa\Im(g) & 0 & \varepsilon^{-1}f \end{pmatrix}$$
$$+ \begin{pmatrix} 2A_0^2\Re(c) & 0 & A_0\Re(d) \\ 2A_0^2\Im(c) & 0 & A_0\Im(d) \\ 0 & 0 & 0 \end{pmatrix} =: M_0 + \hat\sigma M_1 + \hat\sigma^2 M_2.$$

Likewise (see, e.g., [WZ2]), the linearization of the Darcy model (1.18), reduced to constant coefficients, is

$$\begin{aligned} m(\varepsilon,\sigma) &= -\sigma^2 \begin{pmatrix} \Re(a) & -\Im(a) \\ \Im(a) & \Re(a) \end{pmatrix} + i\sigma \begin{pmatrix} -2\kappa\Im(a) & -2\kappa\Re(a) \\ +2\kappa\Re(a) & -2\kappa\Im(a) \end{pmatrix} \\ &\quad + \begin{pmatrix} 2A_0^2\Re(\hat c) & 0 \\ 2A_0^2\Im(\hat c) & 0 \end{pmatrix} =: m_0 + \hat\sigma m_1 + \hat\sigma^2 m_2. \end{aligned} \tag{3.2}$$

Note that the linearization of (1.18) is the linearization of the first two equations of the full model (mcGL) subject to relation

$$B(A) = B_0 + f^{-1}h|A_0|^2 - f^{-1}h|A|^2 = B_0 + f^{-1}h|A_0|^2 - f^{-1}h(U^2+V^2)$$

of (1.17). By the chain rule, this is equivalent to

$$\begin{pmatrix} \mathrm{Id}_2 & 0 \end{pmatrix} L(\varepsilon,\sigma,\hat x) \begin{pmatrix} \mathrm{Id}_2 \\ dB/d(U,V) \end{pmatrix}, \qquad dB/d(U,V)|_{(\bar U,\bar V)} = \begin{pmatrix} -2\bar U f^{-1}h & -2\bar V f^{-1}h \end{pmatrix},$$

where L is the linearization about the periodic wave in the full model (mcGL). As the same exponential coordinate transformation taking L to constant-coefficient takes $dB/d(U,V)$ to a constant-coefficient right multiplier $\begin{pmatrix} \mathrm{Id}_2 & N \end{pmatrix}$ canceling the singularity in the third equation of the full system (2.3), we may conclude that

$$m(\varepsilon,\sigma) = \begin{pmatrix} \mathrm{Id}_2 & 0 \end{pmatrix} M(\varepsilon,\sigma) \begin{pmatrix} \mathrm{Id}_2 \\ N \end{pmatrix}, \quad \text{with } N = \begin{pmatrix} -f^{-1}2A_0h & 0 \end{pmatrix}, \tag{3.3}$$

or

$$\begin{aligned} m(\varepsilon,\sigma) &= -\sigma^2 \begin{pmatrix} \Re(a) & -\Im(a) \\ \Im(a) & \Re(a) \end{pmatrix} + i\sigma \begin{pmatrix} -2\kappa\Im(a) & -2\kappa\Re(a) \\ +2\kappa\Re(a) & -2\kappa\Im(a) \end{pmatrix} \\ &\quad + \begin{pmatrix} 2A_0^2\Re(c) - f^{-1}2A_0^2h\Re(d) & 0 \\ 2A_0^2\Im(c) - f^{-1}2A_0^2h\Im(d) & 0 \end{pmatrix}. \end{aligned} \tag{3.4}$$

Alternatively, comparing (3.4) against (3.2), and using relation $\hat c = c - df^{-1}h$ from (1.19), we may verify (3.3) by direct computation.

3.2. **Matrix perturbation expansion.** We complete our study by a matrix perturbation analysis of M. Specifically, taking a matrix perturbation point of view, we observe that the change of coordinates $M \to S^{-1}MS$ with

$$S = \begin{pmatrix} \mathrm{Id}_2 & 0 \\ N & \mathrm{Id}_m \end{pmatrix} \tag{3.5}$$

block diagonalizes the singular portion $\varepsilon^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2A_0 h & 0 & f \end{pmatrix}$ of M, giving

$$S^{-1}M(\varepsilon,\sigma)S = \begin{pmatrix} m(\varepsilon,\sigma) & 0 \\ 0_2 & i\sigma\varepsilon^{-1} f \end{pmatrix} + \begin{pmatrix} 0_2 & O(1) \\ O(1) & O(1) \end{pmatrix}. \tag{3.6}$$

Proposition 3.1. *Let* $\det f \neq 0$*. Then, for* $|\sigma| \in [1/C, C]$*, for any fixed* $C > 0$*, and* ε *sufficiently small,* $\operatorname{spec} M(\varepsilon,\sigma)$ *consists of* m *eigenvalues lying within* $o(\varepsilon^{-1})$ *of* $\varepsilon^{-1} \operatorname{spec} f$ *together with* 2 *eigenvalues lying within* $o(1)$ *of Darcy eigenvalues* $\operatorname{spec}(m)$*.*

Proof. The first, block-diagonal term of approximate block-diagonalization (3.6) under the assumption $\det f \neq 0$ has spectral separation of order $\sigma/\varepsilon \gtrsim \varepsilon^{-1}$ between the upper left and lower right blocks. By standard matrix perturbation theory [K, MZ, PZ], it follows that that there exists a further *exact* near-identity diagonalizer

$$T := \begin{pmatrix} \mathrm{Id}_2 & \theta_1 \\ \theta_2 & \mathrm{Id} \end{pmatrix} \tag{3.7}$$

with $|\theta_j|$ bounded by the size $O(1)$ of off-diagonal blocks divided by the spectral separation, or $|\theta_j| = O(\varepsilon)$. This gives

$$T^{-1}S^{-1}M(\varepsilon,\sigma)ST = \begin{pmatrix} m(\varepsilon,\sigma) + O(\varepsilon) & 0 \\ 0_2 & \varepsilon^{-1} i\sigma f + O(1) \end{pmatrix}, \tag{3.8}$$

whence the result follows by continuity of spectra under matrix perturbation. □

Corollary 3.2. *For* $\hat{c} \neq 0$*,* $\kappa^2 \neq \kappa^2_{Dstab}$*, and* $\det f \neq 0$*, stability of the Darcy system* (1.18)–(1.19) *is* necessary *for stability of the full model* (mcGL) *for* ε *sufficiently small.*

Proof. For $\hat{c} \neq 0$, $\kappa^2 \neq \kappa^2_{Dstab}$, failure of stability of the Darcy system (1.18)–(1.19) implies that some Darcy eigenvalue in $\operatorname{spec}(m)$ must take on a strictly positive real part for some $|\sigma_*| > 0$. Taking $C > 0$ large enough in Proposition 3.1 that $\sigma_* \in [1/C, C]$, we find for ε small enough that the corresponding eigenvalue of $M(\varepsilon, \sigma_*)$ must also have strictly positive real part, by continuity. Thus, Darcy stability is necessary for full stability, by contradiction. □

Remark 3.3. In the above argument, we used crucially that $(\sigma/\varepsilon)f$ have eigenvalues of norm $\gtrsim \varepsilon^{-1}$ assuming $\det f \neq 0$, or equivalently $\sigma \gtrsim 1$, and also uniform boundedness of all terms not involving ε^{-1}, in particular σ^2. Thus, the restriction $\sigma \in [1/C, C]$ is sharp. Note that this region in which the Darcy model is relevant is disjoint from the region $|\sigma| \leq \varepsilon/C$ for which the Taylor expansion leading to full Eckhaus conditions is valid.

Remark 3.4. Recalling that the Eckhaus stability condition $\mu_t^0 < 0$ in the translational mode, already shown to be necessary for stability in Section 2, agrees with the Darcy condition for stability of *its* translational mode, we see that the new information contained in Corollary 3.2 is precisely the condition for stability of the order-one Darcy mode: $\Re(\hat{c}) < 0$.

3.3. **Refinement in the scalar case.** Let us show that Proposition 3.1 can be refined in the scalar case.

Proposition 3.5. *Suppose $\mu_c^0 < 0$. Then $\mu_t = \mu_{D,t} + O(\varepsilon)$ where $\lambda_{D,t}(\hat\sigma) = i\alpha_{D,t}\hat\sigma + \mu_{D,t}\hat\sigma^2$ is the expansion of the neutral eigenvalue of* (3.2)*.*

Proof. From Proposition 2.11, we can conclude that $\Re(\hat b(\kappa^2))$ is an increasing function of κ^2 as (1.6) implies $A_0(\kappa^2)$ is a decreasing function and $\Re(d)h/f < 0$. Moreover, $\hat b(0)$ is given by

$$\hat b(0) = \tilde b - \frac{dh}{f}\frac{\Re(\tilde b)}{\Re(c)} = \tilde b - \frac{dh}{f\Re(c)}\Re(\tilde b).$$

Taking real parts, we find

$$\Re(\hat b(0)) = \Big(1 - \frac{\Re(d)h}{f\Re(c)}\Big)\Re(\tilde b) > 0,$$

by the lower bound on $\Re(d)h/f$ in Proposition 2.11.

Hence the Darcy reduction gives a complex Ginzburg-Landau equation of the form studied in [WZ2], from which we can obtain an expression for $\mu_{D,t}$ by either Kato-style expansion as in [WZ2] and then rearranging or by matched asymptotics as done in Appendix C. In either case, the expression we want is

$$\mu_{D,t} = -\frac{\big(-2\kappa\Im(a) - \alpha_{D,t}\big)^2 + 4\kappa^2\Re(a)^2 + 2A_0^2\big(\Re(a)\Re(\hat c) + \Im(a)\Im(\hat c)\big)}{2A_0^2\Re(\hat c)}, \tag{3.9}$$

with

$$\alpha_{D,t} = -2\kappa\Im(a) + 2\kappa\Re(a)\frac{\Im(\hat c)}{\Re(\hat c)}. \tag{3.10}$$

Comparing (3.9) and (2.56) completes the proof, where we've used the critical observation that the Darcy reduction at the desired frequency has the same amplitude A_0 as the original amplitude system. □

Note that this refinement is still of the form "Darcy stability is necessary but not sufficient" as we did not need $\Re(d)h/f < 0$, only the supercriticality condition $\Re(\hat c) < 0$. To adapt the proof to $\Re(d)h/f > 0$, we note that then $\Re(\hat b)$ is then a decreasing function of κ^2, with $\Re(\hat b(\kappa_E^2)) = \Re(\tilde b) > 0$.

Let us comment that the Darcy reduction at frequency κ_* does not necessarily carry stability information for other frequencies, and in particular the Eckhaus condition for a typical κ_* does not play a significant role. Indeed, for fixed κ_* and assuming $\mu_c^0 < 0$, the existence range for the Ginzburg-Landau equation is smaller than that of the original amplitude system. In fact, by cleverly choosing a, b, c, d, f, h, one can make the existence range for Darcy at small κ_*^2 smaller than the stable range for the original amplitude system.

Remark 3.6. One might call our Darcy reduction "amplitude-adapted" as it perfectly reconstructs the periodic wave it was generated by. The downside of this "amplitude-adapted" Darcy reduction is that it only carries information about that specific wave.

An alternative way to perform the Darcy reduction, which one might call "frequency-adapted", is to choose $\tilde B_0$ to be B_0, so that the reduced Ginzburg-Landau equation is

$$A_{\hat t} = aA_{\hat x\hat x} + \tilde b A + \hat c|A|^2A.$$

This Ginzburg-Landau equation has periodic traveling waves in the same frequency range as the original amplitude system, hence the name "frequency-adapted" as it is generically the only way to preserve the range of frequencies, but the waves generically have different amplitudes compared to the corresponding frequency in the original amplitude system. An entirely similar computation to what we've done so far yields an Eckhaus condition for the "frequency-adapted" Darcy reduction of the form

$$\kappa^2 \le \tilde{\kappa}_S^2 := \frac{2\frac{\Re(\tilde{b})}{\Re(\hat{c})}(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c}))}{4\Re(a)^2(1+\hat{q}^2) + 2\frac{\Re(a)}{\Re(\hat{c})}\big(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})\big)}.$$

We claim under the assumption that $\mu_c^0 < 0$ and the Benjamin-Feir-Newell criterion (BFN) that $\kappa_S^2 \le \tilde{\kappa}_S^2$ for κ_S^2 as in (2.60). Indeed, Proposition 2.11 can be rephrased as $\Re(c) < \Re(\hat{c}) < 0$, and so we readily obtain

$$\Re(c)4\Re(a)^2(1+\hat{q}^2)+2\Re(a)\big(\Im(a)\Im(\hat{c})+\Re(a)\Re(\hat{c})\big) \le \Re(\hat{c})4\Re(a)^2(1+\hat{q}^2)+2\Re(a)\big(\Im(a)\Im(\hat{c})+\Re(a)\Re(\hat{c})\big).$$

Taking the reciprocal then yields

$$\frac{1}{\Re(c)4\Re(a)^2(1+\hat{q}^2) + 2\Re(a)\big(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})\big)} \ge \frac{1}{\Re(\hat{c})4\Re(a)^2(1+\hat{q}^2) + 2\Re(a)\big(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})\big)}.$$

Multiplying both sides by $\Re(\tilde{b})(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})) < 0$ by (BFN) then yields

$$\frac{\Re(\tilde{b})(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c}))}{\Re(c)4\Re(a)^2(1+\hat{q}^2) + 2\Re(a)\big(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})\big)} \le \frac{\Re(\tilde{b})(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c}))}{\Re(\hat{c})4\Re(a)^2(1+\hat{q}^2) + 2\Re(a)\big(\Im(a)\Im(\hat{c}) + \Re(a)\Re(\hat{c})\big)}.$$

As this is a simple rearrangement of $\kappa_S^2 \le \tilde{\kappa}_S^2$, we obtain our desired conclusion. Hence, we conclude that the "frequency-adapted" Darcy reduction also carries necessary but not sufficient stability about *every* wave, however, it is less precise than the stability information of the "amplitude-adapted" Darcy reduction.

4. Sufficiency of Eckhaus conditions, case $m = 1$

Up to now, we have determined asymptotics for the second-order Taylor expansion at the origin of the dispersion relation for the singular (mcGL) model. This gives useful necessary conditions (2.63) for stability of exponential solutions in terms of the signs of the real parts of second-order coefficients μ_t and μ_c. We shall show later that the second-order Taylor expansions for (mcGL) well-approximate those for the key neutral modes of the exact spectrum of the linearized operator around spatially periodic convective Turing patterns, hence these conditions are necessary also for diffusive stability of the full periodic waves. Indeed, the proof relies strongly on the spectral perturbation analysis done above for (mcGL), the exact spectra being shown to coincide to that of a small perturbation of the Fourier symbol analyzed above.

The above study, however, concerns only the radius of convergence of the Taylor series about the origin, which can be seen (see Section 4) to correspond to $|\hat{\sigma}| \le \varepsilon/C$ in the scaling for (mcGL). To obtain *sufficient* conditions for stability, we must show stability for all $\hat{\sigma} \in \mathbb{R}$.

In this section we continue our study, treating the two-parameter matrix perturbation in ε and $\hat{\sigma}$ as a one-parameter perturbation in various regimes, in order to complete all cases and obtain equivalent conditions for stability: specifically, to show that (strict) *asymptotic diffusive stability* (2.62) *is sufficient for stability,* just as (nonstrict) *asymptotic neutral stability* (2.63) *is necessary.*

We start by introducing the rescaled parameter $\check\sigma := \hat\sigma/\varepsilon$, or $\hat\sigma = \varepsilon\check\sigma$, resulting in a desingularized matrix perturbation problem

$$
\begin{aligned}
\lambda \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} = \Bigg(- \check\sigma^2\varepsilon^2 \begin{pmatrix} \Re(a) & -\Im(a) & 0 \\ -\Im(a) & \Re(a) & 0 \\ 2A_0\Re(g) & 2A_0\Im(g) & e_B \end{pmatrix} + i\check\sigma \begin{pmatrix} O(\varepsilon) & O(\varepsilon) & 0 \\ O(\varepsilon) & O(\varepsilon) & 0 \\ 2A_0 h + O(\varepsilon) & 0 & f \end{pmatrix} \\
+ \begin{pmatrix} 2A_0^2\Re(c) & 0 & A_0\Re(d) \\ 2A_0^2\Im(c) & 0 & A_0\Im(d) \\ 0 & 0 & 0 \end{pmatrix} \Bigg) \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix}
\end{aligned}
\tag{4.1}
$$

corresponding to a parabolic singular perturbation of a relaxation system.

4.1. **Case (i) $|\check\sigma| \le 1/C$, $C \gg 1$ ($\hat\sigma \le \varepsilon/C$).** On this region, we consider (2.41) as a family of matrix perturbation problems in $\check\sigma$ parametrized by ε, with coefficients uniformly bounded, and Taylor expand around $\check\sigma = 0$ as described just above. The balancing procedure described in Section 2.2 then yields an analytic expansion in $\check\sigma$ of associated eigenvalues that is uniformly convergent in a small ball with respect to ε sufficiently small.

Comparing to the expansions (2.26), (2.42) in σ derived in Section 2, we see that these are

$$
\begin{aligned}
\hat\lambda_s(\varepsilon, \check\sigma) = 2A_0^2\Re(c) + O(\check\sigma),\ \hat\lambda_t &= \check\alpha_t\check\sigma + \check\mu_t\check\sigma^2 + O(\check\sigma^3), \\
\hat\lambda_c = \check\alpha_c\check\sigma + \check\mu_c\check\sigma^2 + O(\check\sigma^3),
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
\check\alpha_t &= \varepsilon\alpha_t = O(\varepsilon)\,, \\
\check\alpha_c &= \varepsilon\alpha_c = i(f + 2A_0hp) + O(\varepsilon),
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
\check\mu_t &= \varepsilon^2\mu_t = \varepsilon\mu_t^0 + O(1)\,, \\
\check\mu_c &= \varepsilon^2\mu_c = 2A_0ph(f + 2A_0hp) + O(\varepsilon).
\end{aligned}
\tag{4.4}
$$

Thus the stability conditions are sufficient as well as necessary on this region.

4.2. **Case (ii) $1/C \le |\check\sigma| \le C$ ($\varepsilon/C \le |\hat\sigma| \le C\varepsilon$).** On this region, we take a different point of view, considering (4.1) as a compact family of matrix perturbation problems parametrized by

$$1/C \le |\check\sigma| \le C,$$

$C > 0$ arbitrary, with perturbation parameter $\rho := \varepsilon\check\sigma = o(\check\sigma)$, with again all coefficients uniformly bounded. That is, we consider

$$M(\check\sigma) := M_0(\check\sigma) + \rho M_1(\check\sigma) + \rho^2 M_2(\check\sigma)$$

with

$$
\begin{aligned}
M_0 &= \begin{pmatrix} 2A_0^2\Re(c) & 0 & A_0\Re(d) \\ 2A_0^2\Im(c) & 0 & A_0\Im(d) \\ 2A_0hi\check\sigma & 0 & fi\check\sigma \end{pmatrix}, \\
M_1 &= i \begin{pmatrix} -2\kappa\Im(a) & -2\kappa\Re(a) & 0 \\ 2\kappa\Re(a) & -2\kappa\Im(a) & 0 \\ 2A_0\kappa\Im(g) & 0 & 0 \end{pmatrix}, \\
M_2 &= - \begin{pmatrix} \Re(a) & \Im(a) & 0 \\ -\Im(a) & \Re(a) & 0 \\ 2A_0\Re(g) & 2A_0\Im(g) & e_B \end{pmatrix}.
\end{aligned}
\tag{4.5}
$$

Evidently, $\det M_0 \equiv 0$, hence we may factor the characteristic polynomial

$$p(\lambda; \check\sigma) := \det(\lambda - M_0(\check\sigma)) \tag{4.6}$$

as $p(\lambda;\check\sigma) = \lambda q(\lambda;\check\sigma)$, where q is quadratic, namely

$$q(\lambda;\check\sigma) = \lambda^2 + \beta\lambda + \gamma, \tag{4.7}$$

where

$$\beta = -(2A_0^2\Re(c) + fi\check\sigma), \quad \gamma = 2A_0^2 if\check\sigma\Big(\Re(c) - \frac{h\Re(d)}{f}\Big) = 2A_0^2 if\check\sigma\Re(\hat c), \tag{4.8}$$

where, following (1.19), $\hat c := c - \frac{dh}{f}$ is the updated value of c in the Darcy reduction.

Remark 4.1. The factorization (4.7) can also be seen by direct expansion of the determinant along the central column of $M_0(\check\sigma)$.

Let us investigate the appearance of a pure imaginary root $\lambda = i\tau$. This gives from

$$0 = q(i\tau;\check\sigma) = -\tau^2 + \beta i\tau + \gamma$$

the pair of equations

$$\begin{aligned} 0 &= \Re q(i\tau;\check\sigma) = -\tau^2 - \tau\Im(\beta) + \Re(\gamma) = \tau^2 - \tau f\check\sigma, \\ 0 &= \Im q(i\tau;\check\sigma) = \tau\Re(\beta) + \Im(\gamma) = 2A_0^2\Big(-\Re(c)\tau + \check\sigma f\Re(\hat c)\Big). \end{aligned} \tag{4.9}$$

Solving, we obtain for $\tau = 0$ the trivial solution $\tau = 0$, $\check\sigma = 0$, which we have specifically excluded by taking $\check\sigma \neq 0$, or else $\tau = 0$, $\hat c = 0$, which we exclude by the genericity assumption

$$\hat c \neq 0, \tag{4.10}$$

and, otherwise, the linear system

$$\tau = f\check\sigma, \quad \tau = \check\sigma f\frac{\Re(\hat c)}{\Re(c)}, \tag{4.11}$$

which is consistent for $\check\sigma \neq 0$ if and only if

$$\Re(c) = \Re(\hat c) \iff 0 = h\Re(d).$$

We exclude the latter case by the genericity assumption

$$h\Re(d) \neq 0. \tag{4.12}$$

Recall that first-order coupling coefficients are d and h, with the hyperbolic model decoupling if either of these vanish. Thus, failure of (4.12) is related to a degenerate (at least partial) decoupling of the system.

Assuming (4.12), we find that there are no imaginary eigenvalues of q on $1/C \leq |\check\sigma| \leq C$. Thus, by continuity/compactness, there is a uniform spectral gap between $\lambda = 0$ and the remaining two eigenvalues of $p(\cdot;\check\sigma)$. By standard matrix perturbation theory, this yields an analytic branch $\lambda_0(\rho;\check\sigma)$ with $\lambda_0(0;\sigma) = 0$, convergent for $|\rho| = \varepsilon|\check\sigma|$ sufficiently small. By standard matrix perturbation theory, we obtain also continuous expansions in ρ of the remaining two nonzero eigenvalues of M_0.

Stability of large (nonzero real part) eigenvalues is straightforward, at least for $|\check\sigma|$ bounded, since continuity of spectrum is then enough to conclude stability or instability on the whole domain. Thus, the problem reduces to checking stability at the left endpoint $|\check\sigma| = 1/C$, already determined by Taylor series analysis. Stability of the perturbed zero mode requires more care, as, being neutral to lowest order, it will be determined by the second-order term in the Taylor expansion.

4.2.1. *Matched expansion of the zero mode.* As often the case with matrix perturbation theory, the abstract theory (as above) is useful for establishing analyticity, but it is easier to compute coefficients by positing an analytic expansion and finding coefficients by matching terms at successive orders. Namely, defining $\lambda_0(\rho) = \lambda_1\rho + \lambda_2\rho^2 + H.O.T.$, we may expand the characteristic polynomial equation $0 = \det(M(\rho;\breve\sigma) - \lambda_0)$, after factoring out a common factor ρ from the second column, as

(4.13)
$$\begin{aligned}
0 &= \det\begin{pmatrix} 2A_0^2\Re(c) + \rho(-2i\kappa\Im(a) - \lambda_1) & \rho\Im(a) - 2i\kappa\Re(a) & A_0\Re(d) \\ 2A_0^2\Im(c) + 2i\kappa\Re(a)\rho & -2i\kappa\Im(a) - \lambda_1 + \rho(\Re(a) - \lambda_2) & A_0\Im(d) \\ 2A_0 i\kappa\Im(g)\rho + 2A_0 h i\breve\sigma & \rho 2A_0\Im(g) & -\rho\lambda_1 + fi\breve\sigma \end{pmatrix} + O(\rho^2) \\
&= \det\begin{pmatrix} 2A_0^2\Re(c) & -2i\kappa\Re(a) & A_0\Re(d) \\ 2A_0^2\Im(c) & -2i\kappa\Im(a) - \lambda_1 & A_0\Im(d) \\ 2A_0 h i\breve\sigma & 0 & fi\breve\sigma \end{pmatrix} + O(\rho) \\
&= i\breve\sigma\det\begin{pmatrix} 2A_0^2\Re(c) & -2i\kappa\Re(a) & A_0\Re(d) \\ 2A_0^2\Im(c) & -2i\kappa\Im(a) & A_0\Im(d) \\ 2A_0 h & 0 & f \end{pmatrix} - i\breve\sigma\lambda_1\det\begin{pmatrix} 2A_0^2\Re(c) & 0 & A_0\Re(d) \\ 2A_0^2\Im(c) & 1 & A_0\Im(d) \\ 2A_0 h & 0 & f \end{pmatrix} + O(\rho),
\end{aligned}$$

where in the third inequality we have factored out $i\breve\sigma$ from the third row. Thus,

$$\begin{aligned}
\lambda_1 &= i\det\begin{pmatrix} 2A_0^2\Re(c) & -2\kappa\Re(a) & A_0\Re(d) \\ 2A_0^2\Im(c) & -2\kappa\Im(a) & A_0\Im(d) \\ 2A_0 h & 0 & f \end{pmatrix} \Big/ \det\begin{pmatrix} 2A_0^2\Re(c) & 0 & A_0\Re(d) \\ 2A_0^2\Im(c) & 1 & A_0\Im(d) \\ 2A_0 h & 0 & f \end{pmatrix} \\
&= i\Big(-2\kappa\Im(a) + 2\kappa\Re(a)\frac{\Im(\tilde c)}{\Re(\tilde c)}\Big).
\end{aligned} \tag{4.14}$$

We note that λ_1 *is pure imaginary* (as noted earlier by symmetry), *and independent of* $\breve\sigma$.

To determine the second-order coefficient λ_2, we now expand to the next order in (4.13) and match first-order coefficients in ρ. Expanding the first line of (4.13) and collecting first-order terms, we have the term

$$-i\breve\sigma\lambda_2\rho\det\begin{pmatrix} 2A_0^2\Re(c) & 0 & A_0\Re(d) \\ 2A_0^2\Im(c) & 1 & A_0\Im(d) \\ 2A_0 h & 0 & f \end{pmatrix} = -i\breve\sigma\lambda_2\rho 2A_0^2\big(\Re(c)f - h\Re(d)\big),$$

as the only term involving λ_2, similarly as in the previous calculation. The remaining terms can be calculated from the $\mathcal{O}(\rho)$ coefficient in (4.13) after using multilinearity to remove the λ_2 in the central column. In particular, denoting the remaining terms as $\rho\Sigma$, where

(4.15)
$$\Sigma := \frac{\partial}{\partial\rho}\det\begin{pmatrix} 2A_0^2\Re(c) + \rho(-2i\kappa\Im(a) - \lambda_1) & -2i\kappa\Re(a) + \rho\Im(a) & A_0\Re(d) \\ 2A_0^2\Im(c) + 2i\kappa\Re(a)\rho & (-2i\kappa\Im(a) - \lambda_1) + \rho\Re(a) & A_0\Im(d) \\ 2A_0 h i\breve\sigma + 2A_0 i\kappa\Im(g)\rho & 2A_0\Im(g)\rho & fi\breve\sigma - \rho\lambda_1 \end{pmatrix}\Bigg|_{\rho=0}$$

we see that we have

$$\lambda_2 = \frac{i\Sigma}{\breve\sigma 2A_0^2\big(\Re(c)f - h\Re(d)\big)}. \tag{4.16}$$

While the actual expression for Σ in (4.15) is quite complicated, it is important to note that it is *affine* in $\breve\sigma$ as can readily be seen by performing cofactor expansion along the third column. Hence, to determine stability by $\text{sgn}\lambda_2$, we need only compute the sign of $\Im(\Sigma)$.

The expression for $\Im(\Sigma)$ is also rather complicated, but the important point is that it too is affine in $\breve\sigma$, hence, by (4.16), $\Re(\lambda_2)$ is affine in $(1/\breve\sigma)$.

Thus, to check stability, it is sufficient to check only at the endpoints, $|\check\sigma| = 1/C$- where it is already known from the Taylor expansion- and $|\check\sigma| = C$, $C > 0$ arbitrarily large. At the latter boundary, $|\check\sigma|$ large, we may either check by throwing out $O(1/\check\sigma)$ terms in the just-completed computation, or else by comparing to a separate computation for $|\check\sigma| \geq C$, $C > 0$ sufficiently large, to be done in the next steps. As the first possibility is rather complicated, we will follow that latter approach and defer this to the following sections.

Remark 4.2. In the vectorial case $m = 1$, the computations in (4.15)-(4.16) become significantly more complicated, another point in which the scalar and vectorial cases significantly differ. However, the end result is the same, giving a quantity affine in $1/\check\sigma$. Likewise, the computations (4.9)-(4.11), though a bit more complicated, extend to give the same result for $m > 1$. We carry out these computations in detail in Section 5.

4.3. **Case (iii)** $C \leq |\check\sigma| \leq 1/C\varepsilon$ **(**$C\varepsilon \leq |\hat\sigma| \leq 1/C$**).** Here, $|\check\sigma| \gg 1$ but $|\rho| \ll 1$. On this region, reviewing M_0 in (4.5), we see that, assuming as usual invertibility of f, there is now a dominant large eigenvalue $i\check\sigma f$, with associated left and right eigenvectors

$$l = \begin{pmatrix} 2f^{-1}A_0 h & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T. \tag{4.17}$$

Motivated by this observation, we make the global change of coordinates $\tilde M(\sigma) := T^{-1}M(\sigma)T$, where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2f^{-1}A_0 h & 0 & 1 \end{pmatrix}, \tag{4.18}$$

yielding

$$\tilde M(\check\sigma) = \begin{pmatrix} \check M & \theta_1 \\ \theta_2 & if\check\sigma + 2f^{-1}A_0^2 h\Re(d) - e_B\rho^2 \end{pmatrix}, \tag{4.19}$$

with $|\theta_j|, |\rho| = O(1)$ and

$$\check M(\check\sigma) = \begin{pmatrix} 2A_0^2(\Re(c) - \Re(d)f^{-1}h) & 0 \\ 2A_0^2(\Im(c) - \Im(d)f^{-1}h) & 0 \end{pmatrix} + \rho i \begin{pmatrix} -2\kappa\Im(a) & -2\kappa\Re(a) \\ 2\kappa\Re(a) & -2\kappa\Im(a) \end{pmatrix} - \rho^2 \begin{pmatrix} \Re(a) & \Im(a) \\ -\Im(a) & \Re(a) \end{pmatrix}. \tag{4.20}$$

Here, the lower righthand entry is dominated by the $\check\sigma$ order term $if\check\sigma \gg 1$, whereas remaining terms are $O(1)$; in particular, there is separation of order $|\check\sigma|$ between this entry and the spectra of the upper lefthand block, and off-diagonal blocks of order one. Thus, there exists an exact diagonalizing transformation of form $S = \mathrm{Id} + O(1/|\check\sigma|)$ showing that the real part of the largest eigenvalue is $2f^{-1}A_0\Re(d) + o(1)$, which has sign to first order independent of $\check\sigma$. We have from the analysis of case (ii) that this largest eigenvalue must have negative real part at the innermost endpoint $|\check\sigma| = C$, hence

$$2f^{-1}A_0^2 h\Re(d) < 0 \text{ and therefore also } 2f^{-1}A_0^2 h\Re(d) - e_B\rho^2 < 0,$$

yielding by exact diagonalization that this eigenvalue is stable on the entire $|\check\sigma|$-interval.

On the complementary- i.e., upper left- exactly diagonal block, we obtain, similarly an $O(1/|\check\sigma)$ perturbation of (4.20). But, (4.20) may be recognized as the linearized Darcy system, with $\hat\sigma = \varepsilon\check\sigma$ replaced by ρ, on the interval $C\varepsilon \leq |\rho| \leq C$. By standard analysis of the Darcy system, the associated eigenvalues are to leading order $2A_0^2\Re(\hat c)$ and $\mu_t^0\rho^2$. By comparison at the boundary $|\check\sigma| = C$, the former must have negative real part, as one of the $m+1$ stable order-one eigenvalues. We note that we have just verified indirectly the fact shown by direct computation for the scalar case $m = 1$ in Section 3 that the stability conditions imply $\Re(\hat c) < 0$. Our abstract argument here has the advantage that it generalizes to the vector case $m > 1$.

As the latter has $\Re\mu_t < 0$ by assumption/agreement of translational expansions for Darcy and full model, we may conclude that the reduced Darcy system is *stable* for all $\rho \in \mathbb{R}$, by standard stability theory for complex Ginzburg-Landau as may be obtained readily by the quadratic formula: in particular, on the $|\rho| \ll 1$ region under consideration, where it may be obtained by inspection of the second-order Taylor series.

However, the order $|\rho|^2$ perturbation could change sign under $o(1)$ perturbation, hence must be treated differently. Fortunately, having already determined that the two other eigenvalues are order 1 and $|\hat\sigma| \gg 1$, we have a spectral separation of order one between those and the order $|\rho|^2$ eigenvalue, hence can repeat the argument of case (ii) to conclude that the analytic expansion in ρ deduced there remains valid up to $|\rho| \ll 1$, and so the coefficient of ρ^2 in that expansion is affine in $\check\sigma$ as already determined, with domain of validity $\varepsilon/C \le |\check\sigma| \le 1/C\varepsilon$ including both those of cases (ii) and (iii), hence this eigenvalue is stable iff and only if it is stable at the endpoints $|\check\sigma| = \varepsilon/C$ and $|\check\sigma| = 1/C\varepsilon$. As observed in the treatment of case (ii), it is stable at the inner endpoint $|\check\sigma| = \varepsilon/C$ as the outer endpoint of region (i) already determined. At the outer endpoint $|\check\sigma| = 1/C\varepsilon$, on the other hand, the associated Darcy eigenvalue has strictly negative value bounded away from zero, which property persists under $o(1)$ perturbation to give stability of the exactly diagonalized eigenvalue. Thus, we may conclude stability also of this smallest, order $|\rho|^2$ eigenvalue.

Note that this in passing gives stability on the regime of case (ii), completing the analysis there.

4.4. **Case (iv) $1/C\varepsilon \le |\check\sigma| \le 1/C\varepsilon^2$ ($1/C \le |\hat\sigma| \le 1/C\varepsilon$).** On this region, $|\rho| \ge 1/C$ and $1 + \rho^2 \ll |\check\sigma|$, hence the lower righthand entry

$$if\check\sigma + 2f^{-1}A_0^2 h\Re(d) - e_B\rho^2$$

of $\tilde M(\check\sigma)$ in (4.19) still dominates all other terms, giving a spectral separation of order $|\check\sigma|$ between this and the upper lefthand diagonal block, with off-diagonal terms θ_j now of order $1 + |\rho|^2$. It follows that there is an analytic exact block diagonalization of form

$$\mathrm{Id} + O((1 + |\rho|^2)/|\check\sigma| = \mathrm{Id} + o(1)$$

yielding in the lower righthand entry a real part $2f^{-1}A_0^2 h\Re(d) + o(1)$ already verified as stable, and in the 2×2 upper lefhand block $o(1)$ perturbations of the eigenvalues of the associated Darcy system. As the later have real parts bounded above by a constant times $-(1 + |\rho|^2)$, hence by a strictly negative constant, this property persists under $o(1)$ perturbation, giving stability of the corresponding exactly diagonalized eigenvalues.

4.5. **Case (v) $1/C\varepsilon^2 \le |\check\sigma| \le C/\varepsilon^2$ ($1/C\varepsilon \le |\hat\sigma| \le C/\varepsilon$).** Defining $\sigma := \varepsilon^2\check\sigma$, and factoring out $1/\varepsilon^2$, we have the matrix perturbation problem $\varepsilon^{-2}(M_0 + O(\varepsilon))$, where

$$M_0 = \begin{pmatrix} -\sigma^2\Re(a) & -\sigma^2\Im(a) & 0 \\ \sigma^2\Im(a) & \sigma^2\Re(a) & 0 \\ 2A_0hi\sigma - \sigma^2 2A_0\Re(g) & -\sigma^2 2A_0\Im(g) & fi\sigma - \sigma^2 e_B \end{pmatrix}, \tag{4.21}$$

where $1/C \le |\sigma| \le C$, i.e., σ varies on a compact range. Evidently, M_0 is lower block-diagonal, with upper lefthand 2×2 block of symmetric part negative definite with spectral gap $\Re(a)\sigma^2$, and lower lefthand block having negative real part of spectral gap $e_B\sigma^2$. Thus, the spectra of M_0 have real part uniformly negative over the range under consideration, and we obtain negativity of the perturbed spectra by continuity of spectra under perturbation. Combining with previous cases, this verifies stability up to $|\check\sigma| \le C/\varepsilon^2$ provided asymptotic diffusive stability (2.62) holds.

Remark 4.3. In the vectorial case $m > 1$, the real part of the spectrum of the lower righthand block $(fi\sigma - \sigma^2 e_B)$ is not determined simply by the real part of the spectrum of e_B. However, $\Re\,\mathrm{spec}(fi\sigma - \sigma^2 e_B) < 0$ by Assumption (1.9), hence by continuity/compactness we still obtain a uniform spectral gap.

4.6. **Case (vi)** $|\check\sigma| \geq C/\varepsilon^2$ ($|\hat\sigma| \geq C/\varepsilon$). In this case, the order $\rho^2 = \check\sigma^2\varepsilon^2$ terms dominate order $\check\sigma$ and other terms, and stability follows in straightforward fashion from parabolicity of the truncated system. We omit the details of this standard argument; see, e.g., [MZ].

4.7. **Final result.** Combining the results of Sections 4.1-4.2, we have the following simple condition for stability, analogous to those for the standard (nonsingular) complex Ginzburg-Landau system.

Proposition 4.4. *Assuming the generic conditions of supercriticality* (1.7)*, noncharacteristicity of effective flux* (2.40)*, nontrivial Jordan structure* (2.28)*, and nonvanishing of* $\Re\mu_t^0, \Re\mu_c^0$ *in* (2.44)*, asymptotic diffusive stability* (2.62)*, or* $\Re\mu_t^0, \Re\mu_c^0 > 0$ *with* μ_j^0 *as in* (2.44)*, is* necessary and sufficient *for diffusive stability* (2.5) *in the sense of Schneider of periodic (exponential) solutions* (1.1) *of* (mcGL) *with* $m = 1$*, for* $0 < \varepsilon \leq \varepsilon_0$ *sufficiently small, where* ε_0 *is uniform on compact parameter sets satisfying the above assumptions.*

Proof. For fixed model parameters, the result follows for $\varepsilon > 0$ sufficiently small, by the arguments of cases (i)-(vi) above, where the upper bound ε_0 needed for ε depends only on lower bounds for the various quantities assumed to be nonvanishing. As these quantities are continuous, their minima are uniformly bounded from below on compact parameter sets where they do not vanish, and so $\varepsilon_0 > 0$ may be chosen uniformly for compact parameter sets on which our hypotheses hold. □

Coefficients μ_t and μ_t are explicitly computable, giving simple necessary and sufficient conditions for stability on a par with those for the classical (nonsingular) complex Ginzburg-Landau equation.

Remark 4.5. The above argument is reminiscent of multi-parameter spectral perturbation computations carried out in [JNRZ2, §4] and [BJNRZ, §2], in which, similarly, stability in successive regions is related back ultimately to stability of a Taylor expansion about the origin. In all three of these cases, there is an interesting analogy to relaxation systems and the studies of Kawashima, Shizuta, Zeng, and others [ShK, Ze]. For other examples of multi-parameter expansion as related to spectral stability, see, e.g., [PZ, BHZ, FS1, FS2].

5. Extension to m conservation laws

As noted earlier, the arguments in Section 2 for the scalar case $m = 1$ extend for the most part word for word to the vector case $m > 1$, with the various symbols now representing vectors and matrices rather than scalars, modulo the details indicated in Remarks 4.3, 2.7, and 4.2. We shall therefore not repeat the full argument here, but only supply the specific computations in cases (i) and (ii) that are needed to treat the aspects in which the system case requires further analysis beyond what was given for the scalar case.

5.1. **Region (i): Taylor expansion for vectorial case.** There are two main steps in adapting our proof of sufficiency in case (i). The first is to carry out the Kato-style expansions for $\mu_{c,i}$ from the m large eigenvalues and then to use matched asymptotics to obtain μ_t from the small eigenvalue.

We start as before with the preliminary diagonalization. First, we define a vectorized p, thought of as a row vector, by

$$p := -\frac{\Re(d)}{2A_0\Re(c)},$$

while keeping the same value of q as the in the scalar case. We further denote by e_i the standard basis, regarded as column vectors, of $\mathbb{R}^m$. We will let 0 denote an appropriately sized array whose entries are all 0. Then our left and right (generalized) eigenvectors become

$$L_s = \begin{pmatrix} 1 & 0 & -p \end{pmatrix}, \qquad L_t = \begin{pmatrix} q & 1 & -qp \end{pmatrix}, \qquad L_{c,i} = \begin{pmatrix} 0 & 0 & e_i^T \end{pmatrix}, \tag{5.1}$$

and

$$R_s = \begin{pmatrix} 1 \\ -q \\ 0 \end{pmatrix}, \qquad R_t = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad R_{c,i} = \begin{pmatrix} p_i \\ 0 \\ e_i \end{pmatrix}, \tag{5.2}$$

where p_i denotes the i-th entry of p. From here, a similar computation to the $m = 1$ case shows that this basis block diagonalizes M_0, with $\hat{M}_0$ a matrix of the form

$$\hat{M}_0 = \begin{pmatrix} 0 & A_0(\Im(d) + q\Re(d)) \\ 0 & 0 \end{pmatrix}. \tag{5.3}$$

In the vector case, the matrix M_1 is given by

$$M_1 = i \begin{pmatrix} -2\kappa\Im(a) & -2\kappa\Re(a) & 0 \\ 2\kappa\Re(a) & -2\kappa\Im(a) & 0 \\ 2A_0 h\varepsilon^{-1} + 2A_0\kappa\Im(g) & 0 & \varepsilon^{-1} f \end{pmatrix}.$$

Computing the action of M_1 on the left eigenvectors yields

$$\begin{aligned} L_s M_1 &= i \begin{pmatrix} -2\kappa\Im(a) - 2A_0 p\big(\varepsilon^{-1}h + \kappa\Im(g)\big) & -2\kappa\Re(a) & -\varepsilon^{-1}pf \end{pmatrix}, \\ L_t M_1 &= i \begin{pmatrix} -2\kappa\Im(a)q + 2\kappa\Re(a) - 2A_0 qp\big(\varepsilon^{-1}h + \kappa\Im(g)\big) & -2\kappa\Re(a)q - 2\kappa\Im(a) & -\varepsilon^{-1}qpf \end{pmatrix}, \\ L_{c,i} M_1 &= i \begin{pmatrix} 2A_0 h_i\varepsilon^{-1} + 2\kappa A_0\Im(g_i) & 0 & e_i^T f \end{pmatrix}. \end{aligned}$$

We can then readily obtain the expression for M_1 in the eigenbasis as

$$M_1 = \begin{pmatrix} m_1 & s_1 \\ s_2 & \hat{M}_1 \end{pmatrix},$$

with m_1 scalar, s_1 an $m+1$-row vector, s_2 an $m+1$-column vector, and $\hat{M}_1$ an $m+1 \times m+1$ matrix. More specifically,

$$\begin{aligned} m_1 &= L_s M_1 R_s = -\varepsilon^{-1} 2A_0 \vec{p}\vec{h} i + O(1), \\ s_1 &= L_s M_1 \begin{pmatrix} R_t & R_{c,i} \end{pmatrix} = i \begin{pmatrix} -2\kappa\Re(a) & -\varepsilon^{-1}p(f + 2A_0 hp) \end{pmatrix} + i \begin{pmatrix} 0 & O(1) \end{pmatrix}, \\ s_2 &= \begin{pmatrix} L_t \\ L_{c,i} \end{pmatrix} M_1 R_s = i \begin{pmatrix} -2\varepsilon^{-1}A_0 qph + O(1) \\ 2A_0 h\varepsilon^{-1} \end{pmatrix}, \\ \hat{M}_1 &= \begin{pmatrix} L_t \\ L_{c,i} \end{pmatrix} M_1 \begin{pmatrix} R_t & R_{c,i} \end{pmatrix} = i \begin{pmatrix} -2\kappa\Re(a)q - 2\kappa\Im(a) & -\varepsilon^{-1}qp(f + 2A_0 hp) + O(1) \\ 0 & \varepsilon^{-1}(f + 2A_0 hp) + O(1) \end{pmatrix}. \end{aligned} \tag{5.4}$$

We can perform a similar first order diagonalization, by choosing

$$\mathcal{T}(\hat{\sigma}, \varepsilon) = \begin{pmatrix} 1 & 0 \\ \hat{\sigma}t_2 & Id_{m+1} \end{pmatrix} \begin{pmatrix} 1 & \hat{\sigma}t_1 \\ 0 & Id_{m+1} \end{pmatrix},$$

with corresponding inverse

$$\mathcal{T}(\hat{\sigma}, \varepsilon)^{-1} = \begin{pmatrix} 1 & -\hat{\sigma}t_1 \\ 0 & Id_{m+1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\hat{\sigma}t_2 & Id_{m+1} \end{pmatrix}.$$

Setting $N(\varepsilon, \hat{\sigma}) = \mathcal{T} M(\varepsilon, \hat{\sigma})\mathcal{T}^{-1}$ as before, we obtain

$$N(\varepsilon, \hat{\sigma}) = N_0 + \hat{\sigma}N_1(\varepsilon) + \hat{\sigma}^2 N_2(\varepsilon) + \hat{\sigma}^3 N_3(\varepsilon) + O(\hat{\sigma}^4),$$

with $N_0 = M_0$, and N_1 given in block form

$$N_1 = \begin{pmatrix} m_1 & s_1 + t_1(\hat{M}_0 - m_0) \\ s_2 - (\hat{M}_0 - m_0)t_2 & \hat{M}_1 \end{pmatrix},$$

from which we discover that choosing

$$t_1 = -s_1(\hat{M}_0 - m_0)^{-1}, \qquad t_2 = (\hat{M}_0 - m_0)^{-1}s_2, \tag{5.5}$$

makes N_1 block diagonal. Under this choice, we find that the coefficient of $\hat{\sigma}^2$, N_2, is of the form

$$
\begin{aligned}
N_2 = -M_2 &+ \begin{pmatrix} 0 & t_1 \\ t_2 & 0 \end{pmatrix} M_1 + M_1 \begin{pmatrix} 0 & -t_1 \\ -t_2 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & t_1 \\ t_2 & 0 \end{pmatrix} M_0 \begin{pmatrix} 0 & -t_1 \\ -t_2 & 0 \end{pmatrix} + M_0 \begin{pmatrix} t_1 t_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & t_2 t_1 \end{pmatrix} M_0.
\end{aligned}
\tag{5.6}
$$

As M_2 and the left/right eigenvectors are all $O(1)$, we see that to leading order, N_2 is given by

$$N_2 = \begin{pmatrix} 0 & t_1 \\ t_2 & 0 \end{pmatrix} M_1 + M_1 \begin{pmatrix} 0 & -t_1 \\ -t_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & t_1 \\ t_2 & 0 \end{pmatrix} M_0 \begin{pmatrix} 0 & -t_1 \\ -t_2 & 0 \end{pmatrix} + M_0 \begin{pmatrix} t_1 t_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & t_2 t_1 \end{pmatrix} M_0 + O(1).$$

Computing the requisite matrix products and zooming in on the bottom $m+1 \times m+1$ block, we find

$$\hat{N}_2 = t_2 s_1 - s_2 t_1 + t_2 t_1 (\hat{M}_0 - m_0) + O(1).$$

Plugging in (5.5), we then find that

$$\hat{N}_2 = -s_2 t_1 + O(1). \tag{5.7}$$

We observe that, as before, $\hat{M}_0^2 = 0$, and so $(\hat{M}_0 - m_0)^{-1}$ is given by

$$(\hat{M}_0 - m_0)^{-1} = -\frac{1}{m_0}\Big(Id + \frac{1}{m_0}\hat{M}_0\Big). \tag{5.8}$$

In particular, we discover that

$$t_1 = -s_1(\hat{M}_0 - m_0)^{-1} = \frac{1}{m_0} s_1 + \frac{4A_0 i\kappa \Re(a)}{m_0^2}\begin{pmatrix} 0 & \Im(d) + q\Re(d) \end{pmatrix}. \tag{5.9}$$

So to leading order, $\hat{N}_2$ is $-m_0^{-1} s_2 s_1$. Expanding (5.7) out, we get

$$\hat{N}_2 = \frac{1}{m_0}\begin{pmatrix} 4\kappa\Re(a)A_0 qph\varepsilon^{-1} & 2A_0 qphp\big(f + 2A_0 hp\big)\varepsilon^{-2} \\ -4\kappa\Re(a)A_0 h\varepsilon^{-1} & -2A_0 hp(f + 2A_0 hp)\varepsilon^{-2} \end{pmatrix} + H.O.T. \tag{5.10}$$

The final piece of information we need before we "balance" the Jordan block away is $\hat{N}_3$, the bottom $m+1 \times m+1$ block of N_3. We begin by collecting cubic terms in our expansion to get

$$N_3 = M_1 \begin{pmatrix} t_1 t_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & t_2 t_1 \end{pmatrix} M_1 + \begin{pmatrix} 0 & t_1 \\ t_2 & 0 \end{pmatrix} M_1 \begin{pmatrix} 0 & -t_1 \\ -t_2 & 0 \end{pmatrix} + H.O.T.$$

From this, we can extract the bottom right block as

$$\hat{N}_3 = t_2 t_1 \hat{M}_1 - m_1 t_2 t_1 + H.O.T. = t_2 t_1(\hat{M}_1 - m_1) + H.O.T. \tag{5.11}$$

We observe from (5.3) and (5.4) that the reduced spectral problem

$$\hat{N}(\hat{\sigma}, \varepsilon) = \hat{M}_0 + \sigma \hat{M}_1 + \hat{\sigma}^2 \hat{N}_2 + \hat{\sigma}^3 \hat{N}_3 + O(\hat{\sigma}^4),$$

is, to first order in $\hat{\sigma}$, upper block triangular. We then observe that we only need the bottom left block of $\hat{N}_3$ for our balancing transformation. We observe that the first column of $t_2 t_1$ is $O(\varepsilon^{-1})$, as the first entry of t_1 is $O(1)$ and $t_2 = O(\varepsilon^{-1})$. In particular, since $\hat{M}_1$ is also upper block triangular, we find that the bottom left block of $\hat{N}_3$ is $O(\varepsilon^{-2})$, compared to the expected order of $O(\varepsilon^{-3})$.

In order to find the bottom left block of $\hat{N}_3$, we begin by observing that

$$t_2 = -\frac{i}{m_0}\begin{pmatrix} O(\varepsilon^{-1}) \\ 2A_0 h\varepsilon^{-1} + O(1) \end{pmatrix}.$$

Hence, we find that

$$\hat{N}_3 = -\frac{i}{m_0^2}\begin{pmatrix} O(\varepsilon^{-1}) \\ 2A_0 h\varepsilon^{-1} \end{pmatrix}\begin{pmatrix} 2\kappa\Re(a) & O(\varepsilon^{-1}) \end{pmatrix}\begin{pmatrix} 2A_0 ph\varepsilon^{-1} & -\varepsilon^{-1} qpf \\ 0 & \varepsilon^{-1}(f + 2A_0 hp) + \varepsilon^{-1} 2A_0 ph Id_m \end{pmatrix} + H.O.T.$$

Computing the above products, we find

$$\hat{N}_3 = -\frac{i}{m_0^2}\begin{pmatrix} O(\varepsilon^{-2}) & O(\varepsilon^{-3}) \\ 8\varepsilon^{-2}\kappa\Re(a)A_0^2hph + O(\varepsilon^{-1}) & O(\varepsilon^{-3})\end{pmatrix}. \tag{5.12}$$

We now "balance" $\hat{N}(\varepsilon,\hat{\sigma})$ by sending $\hat{N}(\varepsilon,\hat{\sigma}) \to O(\varepsilon,\hat{\sigma}) := \mathcal{S}(\hat{\sigma})\hat{N}(\varepsilon,\sigma)\mathcal{S}(\sigma)^{-1}$ with $\mathcal{S}(\hat{\sigma})$ given by

$$\mathcal{S}(\hat{\sigma}) := \begin{pmatrix} i\hat{\sigma} & 0 \\ 0 & Id_m \end{pmatrix}.$$

Expanding O in powers of $\hat{\sigma}$ as $O(\varepsilon,\hat{\sigma}) = \hat{\sigma}O_1 + \hat{\sigma}^2 O_2 + O(\hat{\sigma}^3)$, we obtain

$$\begin{aligned} O(\varepsilon,\hat{\sigma}) = i\hat{\sigma}&\begin{pmatrix} -2\kappa\big(\Im(a) + q\Re(a)\big) & A_0\big(\Im(d) + q\Re(d)\big) \\ \frac{4\kappa A_0\Re(a)}{m_0}h\varepsilon^{-1} + O(1) & \varepsilon^{-1}(f + 2A_0hp) + O(1)\end{pmatrix} \\ &+ \hat{\sigma}^2\begin{pmatrix} \frac{4\kappa\Re(a)A_0qph}{m_0}\varepsilon^{-1} + O(1) & \varepsilon^{-1}qp(f+2A_0hp) + O(1) \\ -\frac{8\kappa\Re(a)A_0^2hph}{m_0^2}\varepsilon^{-2} + O(\varepsilon^{-1}) & -\frac{2A_0}{m_0}hp(f+2A_0hp)\varepsilon^{-2} + O(\varepsilon^{-1})\end{pmatrix} + O(\hat{\sigma}^3). \end{aligned} \tag{5.13}$$

For the purposes of analyticity of the dispersion relations, we require the reduced convection matrix $f + 2A_0hp$ to be invertible, with simple eigenvalues. Let us recall that the definition of p implies that the reduced convection matrix is independent of κ as $p = -\Re(d)/(2A_0\Re(c))$. Under this assumption we obtain $m+1$ analytic in $\hat{\sigma}$ dispersion relations which we denote λ_t and $\lambda_{c,i}$, $i = 1, ...m$, admitting the expansions

$$\begin{aligned} \lambda_t(\varepsilon,\hat{\sigma}) &= i\alpha_t\hat{\sigma} + \mu_t\hat{\sigma}^2 + O(\hat{\sigma}^3), \\ \lambda_{c,i}(\varepsilon,\hat{\sigma}) &= i\alpha_{c,i}\hat{\sigma} + \mu_{c,i}\hat{\sigma}^2 + (\hat{\sigma}^3), \end{aligned} \tag{5.14}$$

with $\alpha_{c,i}$ admitting the expansion

$$\alpha_{c,i} = \varepsilon^{-1}\alpha_{c,i}^0 + O(1), \tag{5.15}$$

where $\alpha_{c,i}^0 \in \operatorname{spec}(f + 2A_0hp)$.

We note that α_t is real-valued by standard matrix perturbation arguments (as in the scalar case). However, in the vector case, the $\alpha_{c,i}$ are not necessarily real-valued, hence we require the additional, first-order stability condition (1.23) that $i\alpha_{c,i}$ are pure imaginary. If there are $\alpha_{c,i}$ with with nonzero real part, corresponding to a conjugate pair of eigenvalues of the effective flux matrix $(f+2A_0hp)$, then stability fails in the first-order term of the spectral expansion, for $\hat{\sigma}$ of appropriate sign. If on the other hand all $\alpha_{c,i}$ are real and distinct, then we may conclude by symmetry that not only the leading order contribution, but the entire first-order terms of these eigenvalues are pure imaginary, and thus stability holds (neutrally!) at first order (see Remark 2.13). Thus, we require that the reduced convection matrix $f + 2A_0hp$ be strictly hyperbolic, and noncharacteristic. Again, this makes very much sense from the point of view of hyperbolic relaxation on systems, as the effective flux is none other than the convection matrix for the associated Chapman-Enskog equilibrium system, in which ε^{-1} term is required to be in equilibrium, or vanishing to lowest order (see Remark 2.16).

To evaluate the $\mu_{c,i}$, we let r_i and ℓ_i be the right and left eigenvectors of $f + 2A_0hp$ respectively, and set

$$r_{c,i} = \begin{pmatrix} 0 \\ r_i \end{pmatrix} + O(\varepsilon), \qquad \ell_{c,i} = \begin{pmatrix} c_i & \ell_i \end{pmatrix} + O(\varepsilon), \tag{5.16}$$

where c_i is a constant whose value is chosen so that $\ell_{c,i}$ is a left eigenvector of O_1. We remark that we will not need the precise value of c_i in order to determine $\mu_{c,i}$ to leading order. To find $\mu_{c,i}$ to leading order, we compute $\ell_{c,i} O_2 r_{c,i}$, where we find

$$\mu_{c,i} = \varepsilon^{-2}\Big(- \frac{2A_0 \alpha^0_{c,i} \ell_i h p r_i}{m_0}\Big) + O(\varepsilon^{-1}), \tag{5.17}$$

or after plugging in the definition of p and m_0,

$$\begin{aligned} \mu_{c,i} &= \varepsilon^{-2} \ell_i \frac{h\Re(d)}{2A_0^2 \Re(c)}\Big(f - h\frac{\Re(d)}{\Re(c)}\Big) r_i + O(\varepsilon^{-1}) \\ &= \varepsilon^{-2} \frac{\alpha^0_{c,i} \ell_i h \Re(d) r_i}{2A_0^2 \Re(c)^2} + O(\varepsilon^{-1}). \end{aligned} \tag{5.18}$$

The other step in adapting Case (i) is to perform the matched asymptotics to find the coefficients α_t and μ_t. As before, Kato-style expansions are much more complicated than matched asymptotics due to α_t and μ_t being lower order than $\alpha_{c,i}$ and $\mu_{c,i}$. In a similar fashion to the $m = 1$ case, we can factor out an $i\hat{\sigma}$ from the last m rows of $M(\varepsilon, \hat{\sigma})$ and a copy of $i\hat{\sigma}$ from the second column, leading to an expansion of the form

$$\det(M(\varepsilon, \hat{\sigma}) - \lambda_t Id_{m+2}) = (i\hat{\sigma})^{m+1}\big(P_0 + i\hat{\sigma} P_1 + H.O.T.\big),$$

with P_0 given by

$$P_0 := \det \begin{pmatrix} 2A_0^2 \Re(c) & -2\kappa \Re(a) & A_0 \Re(d) \\ 2A_0^2 \Im(c) & -2\kappa \Im(a) - \alpha_t & A_0 \Im(d) \\ 2A_0 h \varepsilon^{-1} + 2A_0 \kappa \Im(g) & 2A_0 \Im(g) & \varepsilon^{-1} f - \alpha_t Id_m \end{pmatrix}. \tag{5.19}$$

Let $f_1, ..., f_m$ denote the eigenvalues of f. By our Kato expansion, we know that we can take $\alpha_t = O(1)$, and so the bottom right $m \times m$ block is invertible. Hence, by standard identities for block matrices

$$\begin{aligned} P_0 = \Big(\prod_{i=1}^m \big(\varepsilon^{-1} f_i - \alpha_t\big)\Big) \det \Bigg(& \begin{pmatrix} 2A_0^2 \Re(c) & -2\kappa \Re(a) \\ 2A_0^2 \Im(c) & -2\kappa \Im(a) - \alpha_t \end{pmatrix} \\ & - \begin{pmatrix} A_0 \Re(d) \\ A_0 \Im(d) \end{pmatrix} (\varepsilon^{-1} f - \alpha_t \mathrm{Id}_m)^{-1} \begin{pmatrix} 2A_0 h \varepsilon^{-1} + 2A_0 \kappa \Im(g) & 2A_0 \Im(g) \end{pmatrix} \Bigg). \end{aligned}$$

We note that

$$\big(\varepsilon^{-1} f - \alpha_t Id_m\big)^{-1} = \varepsilon f^{-1} + O(\varepsilon^2),$$

and so $P_0 = 0$ can be approximated by

$$P_0 = \det \Bigg(\begin{pmatrix} 2A_0^2 \Re(c) & -2\kappa \Re(a) \\ 2A_0^2 \Im(c) & -2\kappa \Im(a) - \alpha_t \end{pmatrix} - \begin{pmatrix} A_0 \Re(d) \\ A_0 \Im(d) \end{pmatrix} f^{-1} \begin{pmatrix} 2A_0 h & 0 \end{pmatrix} + H.O.T. \Bigg) = 0.$$

We then conclude that

$$\alpha_t = -2\kappa \Im(a) + 2\kappa \Re(a) \frac{\Re(c) - \Re(d) f^{-1} h}{\Im(c) - \Im(d) f^{-1} h} + O(\varepsilon), \tag{5.20}$$

which matches the scalar case as $\hat{c} = c - d f^{-1} h$ in the systems case.

Turning to μ_t, we find that P_1 is given by

$$
\begin{aligned}
P_1 := &\det\begin{pmatrix} -2\kappa\Im(a)-\alpha_t & -2\kappa\Re(a) & A_0\Re(d) \\ 2\kappa\Re(a) & -2\kappa\Im(a)-\alpha_t & A_0\Im(d) \\ 2A_0\Re(g) & 2A_0\Im(g) & \varepsilon^{-1}f-\alpha_t Id_m \end{pmatrix} \\
&+\det\begin{pmatrix} 2A_0^2\Re(c) & -\Im(a) & A_0\Re(d) \\ 2A_0^2\Im(c) & \Re(a)+\mu_t & A_0\Im(d) \\ 2A_0h\varepsilon^{-1}+2A_0\kappa\Im(g) & 0 & \varepsilon^{-1}f-\alpha_t Id_m \end{pmatrix} \\
&+\sum_{j=1}^{m}\det\begin{pmatrix} 2A_0^2\Re(c) & -2\kappa\Re(a) & \widehat{A_0\Re(d)}_j \\ 2A_0^2\Im(c) & -2\kappa\Im(a)-\alpha_t & \widehat{A_0\Im(d)}_j \\ 2A_0h\varepsilon^{-1}+2A_0\kappa\Im(g) & 2A_0\Im(g) & \mathcal{P}_j \end{pmatrix},
\end{aligned} \tag{5.21}
$$

where for a vector v, $\widehat{v}_j$ denotes the vector v with the j-th entry set to zero, and $\mathcal{P}_j$ denotes the $m\times m$ matrix

$$
\mathcal{P}_j := \begin{pmatrix}
\varepsilon^{-1}f_{11}-\alpha_t & \varepsilon^{-1}f_{12} & \dots & \varepsilon^{-1}f_{1j-1} & [e_B]_{1j} & \varepsilon^{-1}_{1j+1} & \dots\varepsilon^{-1}f_{1m} & \\
\varepsilon^{-1}f_{21} & \varepsilon^{-1}f_{22}-\alpha_t & \dots & \varepsilon^{-1}f_{2j-1} & [e_B]_{2j} & \varepsilon^{-1}_{2j+1} & \dots\varepsilon^{-1}f_{2m} & \\
\vdots & & & & & & & \\
\varepsilon^{-1}f_{j1} & \varepsilon^{-1}f_{j2} & \dots & \varepsilon^{-1}f_{jj-1} & [e_B]_{jj}+\mu_t & \varepsilon^{-1}f_{jj+1} & \dots & \varepsilon^{-1}f_{jm} \\
\vdots & & & & & & & \\
\varepsilon^{-1}f_{m1} & \varepsilon^{-1}f_{m2} & \dots & \varepsilon^{-1}f_{mj-1} & [e_B]_{mj} & \varepsilon^{-1}f_{mj+1} & \dots & \varepsilon^{-1}f_{mm}-\alpha_t
\end{pmatrix}.
$$

In (5.21), we note that the dominant term is given by

$$
\begin{aligned}
&\det\begin{pmatrix} -2\kappa\Im(a)-\alpha_t & -2\kappa\Re(a) & A_0\Re(d) \\ 2\kappa\Re(a) & -2\kappa\Im(a)-\alpha_t & A_0\Im(d) \\ 2A_0\Re(g) & 2A_0\Im(g) & \varepsilon^{-1}f-\alpha_t Id_m \end{pmatrix} \\
&+\det\begin{pmatrix} 2A_0^2\Re(c) & -\Im(a) & A_0\Re(d) \\ 2A_0^2\Im(c) & \Re(a)+\mu_t & A_0\Im(d) \\ 2A_0h\varepsilon^{-1}+2A_0\kappa\Im(g) & 0 & \varepsilon^{-1}f-\alpha_t Id_m \end{pmatrix},
\end{aligned}
$$

as each entry in the sum over j in (5.21) has one less power of ε, most readily seen by taking the j-th entry and performing cofactor expansion along the j-th column. Indeed, we notice that as the first two entries in that column are zero, there are in each cofactor $m-1$ rows of size ε^{-1} where as the first two matrices have m rows of size ε^{-1}. Taking advantage of the same trick as we did for P_0, we observe that

$$
\begin{aligned}
P_1 = &\Big(\prod_{i=1}^{m}\left(\varepsilon^{-1}f_i-\alpha_t\right)\Big)\det\left(\begin{pmatrix} -2\kappa\Im(a)-\alpha_t & -2\kappa\Re(a) \\ 2\kappa\Re(a) & -2\kappa\Im(a)-\alpha_t \end{pmatrix}+O(\varepsilon)\right) \\
&+\Big(\prod_{i=1}^{m}\left(\varepsilon^{-1}f_i-\alpha_t\right)\Big)\det\left(\begin{pmatrix} 2A_0^2\Re(c) & -\Im(a) \\ 2A_0^2\Im(c) & \Re(a)+\mu_t \end{pmatrix}-\begin{pmatrix} A_0\Re(d) \\ A_0\Im(d) \end{pmatrix}f^{-1}\begin{pmatrix} 2A_0h & 0 \end{pmatrix}+O(\varepsilon)\right) \\
&+O(\varepsilon^{m-1}).
\end{aligned}
$$

Setting $P_1=0$, and computing the leading 2×2 determinants and solving for μ_t, we obtain

$$
\mu_t = -\frac{\big(2\kappa\Im(a)+\alpha_t\big)^2+4\Re(a)^2\kappa^2+2A_0^2\big(\Re(a)\Re(\hat{c})+\Im(a)\Im(\hat{c})\big)}{2A_0^2\Re(\hat{c})}+O(\varepsilon), \tag{5.22}
$$

which we remark matches the scalar case.

Remark 5.1. Much like the scalar case, the $\alpha_{c,i}$ and $\mu_{c,i}$ can also be obtained in this manner. However, unlike the scalar case, the procedure is not as simple. Indeed, our trick here was to invert the bottom right block of (5.19), however, if $\alpha_{c,i} \sim \varepsilon^{-1}$, then there is no guarantee of that holding. Instead, we use the invertibility of the top left corner to instead obtain

$$P_0 = 2A_0^2\Re(c)\det\Bigg(\begin{pmatrix} -2\kappa\Im(a) & A_0\Im(d) \\ 2A_0\Im(g) & \varepsilon^{-1}f \end{pmatrix} - \frac{1}{2A_0^2\Re(c)}\begin{pmatrix} 2A_0^2\Im(c) \\ 2A_0h\varepsilon^{-1} + 2A_0\kappa\Im(g) \end{pmatrix}\begin{pmatrix} -2\kappa\Re(a) & A_0\Re(d) \end{pmatrix} - \alpha_{c,i}Id_{m+1}\Bigg).$$

Collecting the leading order terms, we discover

$$P_0 = 2A_0^2\Re(c)\det\left(\begin{pmatrix} O(1) & O(1) \\ O(\varepsilon^{-1}) & \varepsilon^{-1}\Big(f - \frac{h\Re(d)}{\Re(c)}\Big) + O(1) \end{pmatrix} - \alpha_{c,i}Id_{m+1}\right).$$

As P_0 is approximately the determinant of a lower block triangular matrix, we conclude that $\alpha_{c,i}$ is an eigenvalue of $f - h\Re(d)/\Re(c)$ to leading order, matching our earlier conclusion.

Turning to $\mu_{c,i}$, the determinant independent of $\mu_{c,i}$ in (5.21) is to leading order given by

$$\begin{aligned} c_4 &:= \det\begin{pmatrix} -2\kappa\Im(a) - \alpha_{c,i} & -2\kappa\Re(a) & A_0\Re(d) \\ 2\kappa\Re(a) & -2\kappa\Im(a) - \alpha_{c,i} & A_0\Im(d) \\ 2A_0\Re(g) & 2A_0\Im(g) & \varepsilon^{-1}f - \alpha_{c,i}Id_m \end{pmatrix} \\ &= \alpha_{c,i}^2\det\left(\varepsilon^{-1}f - \alpha_{c,i}Id_m\right) + O(\varepsilon^{-(m+1)}). \end{aligned}$$

The first matrix outside of the sum yields a coefficient of $\mu_{c,i}$ of the form

$$\det\begin{pmatrix} 2A_0^2\Re(c) & A_0\Re(d) \\ 2A_0h\varepsilon^{-1} + 2A_0\kappa\Im(g) & \varepsilon^{-1}f - \alpha_{c,i}Id_m \end{pmatrix}.$$

The same block determinant trick that gave us $\alpha_{c,i}$ shows that this determinant is $O(\varepsilon^{-(m-1)})$, which is an acceptable error term for our purposes.

The remaining sum can be seen to give a coefficient of $\mu_{c,i}$ of the form

$$c_3 := -2A_0^2\Re(c)\alpha_{c,i}\operatorname{Tr}\Big(\operatorname{cof}(\varepsilon^{-1}f - \alpha_{c,i}Id_m - \varepsilon^{-1}\frac{h\Re(d)}{\Re(c)}) + H.O.T.\Big),$$

where $\operatorname{cof}(A)$ denotes the cofactor matrix of A. We recall from [DPTZ] that the adjugate matrix, $\operatorname{adj}(A) = \operatorname{cof}(A)^T$, has the same left/right eigenvectors as A, with the eigenvectors (ℓ_i, r_i) having eigenvalue $\prod_{j\neq i}\lambda_j$, where λ_i denotes the eigenvalues of A. In particular, the cofactor matrix has eigenvalues $\prod_{j\neq i}\lambda_j$, where $i = 1, ..., m$, and so by assuming A has a single eigenvalue $\lambda_i = 0$ we have that

$$\operatorname{Tr}\Big(\operatorname{cof}(\varepsilon^{-1}f - \alpha_{c,i}Id_m - \varepsilon^{-1}\frac{h\Re(d)}{\Re(c)}) + H.O.T.\Big) = \prod_{j\neq i}\alpha_{c,j} + H.O.T. \sim \varepsilon^{-(m-1)}.$$

Hence, $c_3 \sim \varepsilon^{-m}$ is the dominant coefficient of $\mu_{c,i}$ in $P_1 = 0$. To make c_4 more closely resemble c_3, we notice that $\varepsilon^{-1}f - \alpha_{c,i}Id_m$ is a rank-one update of $\varepsilon^{-1}f - \alpha_{c,i}Id_m - h\Re(d)/\Re(c)$, and so one can use the corresponding update formula for the determinants to get

$$\det(\varepsilon^{-1}f - \alpha_{c,i}Id_m) = \det\Big(\varepsilon^{-1}f - \frac{h\Re(d)}{\Re(c)} - \alpha_{c,i}Id_m\Big) + \frac{1}{\Re(c)}\Re(d)\operatorname{adj}\left(\varepsilon^{-1}f - \frac{h\Re(d)}{\Re(c)} - \alpha_{c,i}Id_m\right)h.$$

In particular, the leading order component of $\det(\varepsilon^{-1}f - \alpha_{c,i}Id_m)$ is then, after diagonalizing $\mathrm{adj}(f - h\Re(d)/\Re(c))$, given by

$$\det(\varepsilon^{-1}f - \alpha_{c,i}Id_m) = \frac{1}{\Re(c)}\Re(d)r_i\ell_i h\prod_{j\neq i}\alpha_{c,j} + H.O.T.$$

from which we recover the Kato-style formula from $\mu_{c,i} = -c_4/c_3$.

5.2. **Region (ii): nonexistence of imaginary eigenvalues.** The first is to show that M_0 in case (ii) has no pure imaginary eigenvalues other than the simple zero eigenvalue in its kernel. To this end, we note that the zero e-vectors for M_0 are $\ell = (\alpha, 1, \beta), r = (0, 1, 0)^T$, which gives complementary projections

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 0 \\ -\alpha & -\beta \\ 0 & 1 \end{pmatrix}.$$

This gives the reduced matrix

$$\hat{M}_0 := LM_0R = \begin{pmatrix} 2A_0^2\Re(c) & A_0\Re(d) \\ 2A_0h\hat{i\sigma} & f\hat{i\sigma} \end{pmatrix} \tag{5.23}$$

complementary to the kernel of M_0.

What we need to show is that $\hat{M}_0$ has no imaginary eigenvalues $i\tau$ for $\hat{\sigma} \neq 0$, or, equivalently,

$$\Delta(\tau, \check{\sigma}) := \det\begin{pmatrix} 2A_0^2\Re(c) - i\tau & A_0\Re(d) \\ 2A_0h\check{\sigma} & f\check{\sigma} - \tau \end{pmatrix} \text{ has no real roots } (\tau, \check{\sigma}) \text{ for } \check{\sigma} \neq 0. \tag{5.24}$$

Lemma 5.2. *Under nondegeneracy conditions* (5.27) *and* (5.28) *below, along with* $\Re\hat{c} \neq 0$*, condition* (5.24) *holds for all choices of model parameters.*

Proof. Expanding $\Delta = 0$, and setting real and imaginary parts separately to zero, we obtain for the imaginary part

$$0 = \det\begin{pmatrix} -\tau & A_0\Re(d) \\ 0\check{\sigma} & f\check{\sigma} - \tau \end{pmatrix} = \tau\det(f\check{\sigma} - \tau). \tag{5.25}$$

and (factoring out $2A_0$ from the first column and A_0 from the first row)

$$0 = \det\begin{pmatrix} 2A_0^2\Re(c) & A_0\Re(d) \\ 0\check{\sigma} & f\check{\sigma} - \tau \end{pmatrix} = \det\begin{pmatrix} \Re(c) & \Re(d) \\ h\check{\sigma} & f\check{\sigma} - \tau \end{pmatrix} \tag{5.26}$$

for the real part.

From (5.25), either $\tau = 0$, or else $\det(f\check{\sigma} - \tau) = 0$. In the first case, (5.26) simplifies to

$$0 = \check{\sigma}\det\begin{pmatrix} \Re(c) & \Re(d) \\ h & f \end{pmatrix} = \det(f)\Re(\hat{c}),$$

which is excluded by our nondegeneracy condition $\hat{c} \neq 0$, $\det f \neq 0$, and by the assumption $\check{\sigma} \neq 0$.

Thus, we need only examine the case $\det(f\check{\sigma} - \tau) = 0$, or (using $\tau, \check{\sigma} \neq 0$)

$$\tau/\check{\sigma} \in \sigma(f).$$

Taking without loss of generality coordinates such that f is diagonal,

$$f = \mathrm{diag}\{f_1, \dots, f_m\},$$

impose the additional nondegeneracy conditions of "individual coupling":

$$h_j\Re(d_j) \neq 0 \text{ for each } j = 1, \dots, m \tag{5.27}$$

and strict hyperbolicity

$$\text{spec}(f)\text{ simple.} \tag{5.28}$$

Then, taking $\tau/\check\sigma = f_j$, without loss of generality $j = 1$, and substituting in (5.26), we obtain, factoring out $\check\sigma \neq 0$ from the final row,

$$0 = \check\sigma \det \begin{pmatrix} \Re(c) & \Re(d_1) & \Re(d_2) & \dots & \Re(d_m) \\ h_1 & 0 & 0 & \dots & 0 \\ h_2 & 0 & f_2 - f_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ h_m & 0 & \dots & 0 & f_m - f_1 \end{pmatrix} = \check\sigma h_1 \Re(d_1) \det \begin{pmatrix} f_2 - f_1 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & f_m - f_1 \end{pmatrix},$$

giving $0 = \check\sigma h_1 \Re(d_1)\Pi_{j\neq 1}(f_j - f_1)$, which, by $\check\sigma \neq 0$, (5.27), and (5.28), is a contradiction. □

5.3. **Region (ii): affine dependence on $1/\check\sigma$.** The second is to show that the observation for $m = 1$ that (4.16) is affine in $\check\sigma^{-1}$ remains true in the vector case $m > 1$. The same argument as in 4.2 showing the denominator was proportional to $\check\sigma$ is easily adapted to the vectorial case to show that the corresponding denominator is proportional to $\check\sigma^m$ and is thus omitted. Focusing on the numerator and recalling (4.13), we see that the matrix $\tilde M(\rho;\check\sigma)$ whose determinant is in the numerator takes the block form

$$\tilde M(\rho;\check\sigma) = \begin{pmatrix} \tilde M_{11}(\rho) & \tilde M_{12} \\ \tilde M_{21}(\rho;\check\sigma) & \tilde M_{22}(\rho;\check\sigma) \end{pmatrix}, \tag{5.29}$$

where $\tilde M_{12}$ is a $2\times m$ constant matrix, $\tilde M_{21}$ is a $m \times 2$ matrix taking the form

$$\tilde M_{21}(\rho;\check\sigma) = \begin{pmatrix} 2A_0 i\check\sigma h & 0_{m\times 1}\end{pmatrix} + \mathcal{O}(\rho)_{m\times 2}, \tag{5.30}$$

and $\tilde M_{22}$ is an $m \times m$ matrix taking the form

$$\tilde M_{22}(\rho;\check\sigma) = i\check\sigma f + \mathcal{O}(\rho)_{m\times m}. \tag{5.31}$$

In (5.30) and (5.31), the subscripts denote the shape of the error terms. Of particular interest is the order ρ term in the determinant of $\tilde M(\rho;\check\sigma)$. Let us illustrate the computation in the $m = 2$ case, the argument extends naturally to the higher m-case. Each column of $\tilde M(\rho;\check\sigma)$ may be written as

$$\tilde M_i(\rho;\check\sigma) = \tilde M_i^0(\check\sigma) + \rho \tilde M_i^1(\check\sigma). \tag{5.32}$$

Hence by multilinearity of det, we may compute the coefficient ρ in $\det(\tilde M(\rho;\check\sigma))$ as

$$\begin{aligned} \frac{\partial}{\partial\rho}\det \tilde M(\rho;\check\sigma)|_{\rho=0} &= \det\Big(\big[\tilde M_1^1(\check\sigma) \quad \tilde M_2^0(\check\sigma) \quad \tilde M_3^0(\check\sigma) \quad \tilde M_4^0(\check\sigma)\big]\Big) + \dots \\ &\quad + \det\Big(\big[\tilde M_1^0(\check\sigma) \quad \tilde M_2^0(\check\sigma) \quad \tilde M_3^0(\check\sigma) \quad \tilde M_4^1(\check\sigma)\big]\Big), \end{aligned} \tag{5.33}$$

where the $\tilde M_i^j$ are as in (5.32). There are three types of terms in the sum in (5.33), the first column having upper index one, the second column having upper index one, and the final m columns having upper index one. For each summand in (5.33), we use cofactor expansion along the column whose upper index is one, which we now illustrate for the first three columns when $m = 2$. Starting with the first summand, we find

$$\begin{aligned} \det\Big(\big[\tilde M_1^1(\check\sigma) \quad \tilde M_2^0(\check\sigma) \quad \tilde M_3^0(\check\sigma) \quad \tilde M_4^0(\check\sigma)\big]\Big) &= c_{11}\det\begin{pmatrix} * & * & * \\ 0 & i\check\sigma f_{11} & i\check\sigma f_{12} \\ 0 & i\check\sigma f_{21} & i\check\sigma f_{22}\end{pmatrix} \\ - c_{21}\begin{pmatrix} * & * & * \\ 0 & i\check\sigma f_{11} & i\check\sigma f_{12} \\ 0 & i\check\sigma f_{21} & i\check\sigma f_{22}\end{pmatrix} + c_{31}\det\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & i\check\sigma f_{21} & i\check\sigma f_{22}\end{pmatrix} &- c_{41}\det\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & i\check\sigma f_{11} & i\check\sigma f_{12}\end{pmatrix}, \end{aligned} \tag{5.34}$$

where the $*$'s denote suitable entries of $\tilde{M}_{11}$ and $\tilde{M}_{12}$, whose precise values do not matter save that they are independent of $\check{\sigma}$, and the f_{ij} denote the entries of the matrix f. Finally, we have denoted the entries of the vector $\tilde{M}_1^1$ by c_{j1}, for $j = 1, 2, 3, 4$. Observe that we can always factor out exactly one or two powers of $\check{\sigma}$ in (5.34). For the second summand, we find

$$\begin{aligned}
\det\Big(\big[\tilde{M}_1^0(\check{\sigma}) \quad \tilde{M}_2^1(\check{\sigma}) \quad \tilde{M}_3^0(\check{\sigma}) \quad \tilde{M}_4^0(\check{\sigma})\big] \Big) &= c_{12} \det \begin{pmatrix} * & * & * \\ 2A_0 h_1 i\check{\sigma} & i\check{\sigma} f_{11} & i\check{\sigma} f_{12} \\ 2A_0 h_2 i\check{\sigma} & i\check{\sigma} f_{21} & i\check{\sigma} f_{22} \end{pmatrix} \\
- c_{22} \begin{pmatrix} * & * & * \\ 2A_0 h_1 i\check{\sigma} & i\check{\sigma} f_{11} & i\check{\sigma} f_{12} \\ 2A_0 h_2 i\check{\sigma} & i\check{\sigma} f_{21} & i\check{\sigma} f_{22} \end{pmatrix} &+ c_{32} \det \begin{pmatrix} * & * & * \\ * & * & * \\ 2A_0 h_2 i\check{\sigma} & i\check{\sigma} f_{21} & i\check{\sigma} f_{22} \end{pmatrix} \\
- c_{42} \det \begin{pmatrix} * & * & * \\ * & * & * \\ 2A_0 h_1 i\check{\sigma} & i\check{\sigma} f_{11} & i\check{\sigma} f_{12} \end{pmatrix}&.
\end{aligned} \tag{5.35}$$

As with, (5.34), we find that (5.35) is of the form $\check{\sigma}(a\check{\sigma} + b)$ for known complex coefficients a, b. For our final example column, we find

(5.36)

$$\det\Big(\big[\tilde{M}_1^0(\check{\sigma}) \quad \tilde{M}_2^0(\check{\sigma}) \quad \tilde{M}_3^1(\check{\sigma}) \quad \tilde{M}_4^0(\check{\sigma})\big] \Big) = c_{13} \det \begin{pmatrix} * & * & * \\ 2A_0 h_1 i\check{\sigma} & 0 & i\check{\sigma} f_{12} \\ 2A_0 h_2 i\check{\sigma} & 0 & i\check{\sigma} f_{22} \end{pmatrix}$$

$$- c_{23} \begin{pmatrix} * & * & * \\ 2A_0 h_1 i\check{\sigma} & 0 & i\check{\sigma} f_{12} \\ 2A_0 h_2 i\check{\sigma} & 0 & i\check{\sigma} f_{22} \end{pmatrix} + c_{33} \det \begin{pmatrix} * & * & * \\ * & * & * \\ 2A_0 h_2 i\check{\sigma} & 0 & i\check{\sigma} f_{22} \end{pmatrix} - c_{43} \det \begin{pmatrix} * & * & * \\ * & * & * \\ 2A_0 h_1 i\check{\sigma} & 0 & i\check{\sigma} f_{12} \end{pmatrix}.$$

There is an additional simplification of (5.36) coming from the observation that $\tilde{M}_{12}$ is independent of ρ and so c_{13} and c_{23} vanish, leading us to conclude (5.36) is proportional to $\check{\sigma}$. Similar considerations apply to the final term, where the fourth column has upper index one. Hence combining (5.34), (5.35), and (5.36), we find that (5.33) is of the form

$$\frac{\partial}{\partial \rho} \det \tilde{M}(\rho; \check{\sigma})|_{\rho=0} = \check{\sigma}(A\check{\sigma} + B), \tag{5.37}$$

for known complex constants A and B. To extend to the general case, we observe that the analogs of (5.34), (5.35), and (5.36) always have either m-rows proportional to $\check{\sigma}$ or $(m-1)$-rows proportional to $\check{\sigma}$, leading to the overall conclusion that the numerator is of the form $\check{\sigma}^{m-1}(A\check{\sigma} + B)$, which upon division by $\check{\sigma}^m$, gives us the desired conclusion that λ_2 is affine in $\check{\sigma}^{-1}$.

This completes the proof of the second and final postponed computation from Remark 4.2, thereby completing the proof of sufficiency in the vector case $m > 1$.

5.4. **Final result.** With these modifications, we obtain the following vector version of Proposition 4.4.

Definition 5.3. We define *asymptotic diffusive stability* in the vector case as satisfaction of conditions 1.20, 1.25, and (1.23). We define *asymptotic instability* by failure of *asymptotic neutral stability* defined by satisfaction of (1.23) and

$$\Re\mu_t^0 \geq 0,\ \Re\mu_{c,j}^0 \geq 0. \tag{5.38}$$

Proposition 5.4. *Assuming the generic conditions of supercriticality* (1.7)*, hyperbolicity and noncharacteristicity of effective flux* (1.24) *and* (2.40)*, nontrivial Jordan structure* (2.28)*, and nonvanishing of* $\Re\mu_t^0, \Re\mu_c^0$ *in* (2.44)*, asymptotic diffusive stability* (5.3)*, is* necessary and sufficient *for diffusive stability* (2.5) *in the sense of Schneider of periodic (exponential) solutions* (1.1) *of* (mcGL) *for* $m > 1$*, for* $0 < \varepsilon \le \varepsilon_0$ *sufficiently small, where* ε_0 *is uniform on compact parameter sets satisfying the above assumptions.*

6. Rigorous validation

Finally, we discuss the relation between solutions of (mcGL) and their behavior with convective Turing bifurcation in (1.2), as described in [WZ3]. Denote by $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ the decompositions in nonconservative (i.e., the first $n-m$) and conservative (i.e., the last m) coordinates in (1.2), with f, B, R sufficiently smooth, and B strictly parabolic, and let $u(x,t) \equiv u^*$ be a constant, equilibrium solution of (1.2) satisfying the Turing assumptions [WZ3, Hypothesis 1]. Note that parabolicity of B implies parabolicity of e_B and $\Re(a) > 0$ in (mcGL), by the recipe given in [WZ3], verifying our assumptions on (mcGL).

Under [WZ3, Hypothesis 1(2)], we have R_1 full rank with respect to u_1, hence, local to u^*, there is a function $u_1 = \phi^*(u_2)$ for which $R_1(\phi^*(u_2), u_2) \equiv 0$, uniquely determining u_1^* as a function of u_2^*. Hence, by a coordinate shift $u_1 \to u_1 - \phi^*(u_2)$, $u_2 \to u_2 - u_2^0$, where u_2^0 is some base point under consideration, we may assume without loss of generality $u_1^* = 0$ for any such equilibrium state, while u_2^* varies freely within an open set of u_2^0. Linearizing (1.2) about the constant solution u^*, we assume that the dispersion relation of the associated constant-coefficient symbol has strictly negative spectrum for Fourier frequencies $k \neq 0$ for negative values of the bifurcation parameter ν, and at the bifurcaton point $\nu = 0$ there exists simple eigenvalues $\lambda = \pm i\tau$ at $k = \pm k_* \neq 0$ and an m-fold semisimple eiganvalue $\lambda = 0$ at $k = 0$, with all other spectra strictly negative; moreover, we assume that the real part of the spectrum has second-order contact in k with the imaginary axis at $k = 0, \pm k_*$, departing to second order as k is varied from 0 or $\pm k_*$, and first-order contact at $k = \pm k_*$, growing linearly in ν as ν increases through the bifurcation value $\nu = 0$. For the structure assumed in (1.2), these conditions are equivalent to the Turing assumptions of [WZ3].

Then, by the results of [WZ3], there exists a self-consistent and well-posed multi-scale expansion of form (mcGL), (1.3) formally governing small-amplitude solutions for $\nu > 0$ sufficiently small. Moreover, this expansion may be continued to all orders. The coefficients may be determined as described in [WZ3, §3], with linear terms coming from the eigenstructure of the linear dispersion relation considered as a function of k, ν, and B_0; however, the details of this recipe will not concern us here, other than the fact that generic coefficients of (mcGL) induce generic coefficients of (mcGL), so that generic assumptions on (1.2) may be made via assumptions on (mcGL).

6.1. Existence.

We begin by recalling (a slightly refined version of) the rigorous validation result established previously in [WZ3] in the context of existence of periodic traveling waves. Refining a bit the description of [WZ3, Thm. 6.5], we have the following result asserting existence of exact solutions nearby the asymptotic solutions predicted by (mcGL), (1.1).

Theorem 6.1 (Existence of exact solutions)**.** *Under the above-described Turing Hypotheses (encompassing those of* [WZ3]*), with* $\nu = \varepsilon^2$*, for* κ*,* B_0 *lying in any compact subset of domain* (1.8)*, for* $0 < \varepsilon \le \varepsilon_0$ *sufficiently small there exists a smooth family of traveling-wave solution*

$$u(x,t) = \bar{u}^{\varepsilon,\kappa,B_0}(kx + \bar{\Omega}t), \qquad k = k_* + \varepsilon\kappa \tag{6.1}$$

of amplitude $|u| \sim \varepsilon$ *of* (1.2)*, lying within* $O(\varepsilon^2)$ *of approximate solution* (1.3)*,* (1.1) *with* ω *in* (1.1) *replaced by* $\tilde{\omega} = \omega + O(\varepsilon^3)$ *(thus determining* $\bar{\Omega}$ *up to* $O(\varepsilon^3)$*). Moreover, up to translation, these are the unique nontrivial small-amplitude periodic traveling waves of* (1.2)*.*

Proof. The version of this theorem stated in [WZ3] is for (κ, B_0) sufficiently small. Since choice of the center B^* is arbitrary, that there is no loss of generality in fixing $B_0 \equiv 0$, so long as all estimates are uniform, so that small B_0 is no real restriction. Validation on the full existence domain (1.8), along with the additional detail given here on the speed of the wave, then follows as in [WZ1, Thm.

1.2] for the case without conservation laws. We omit the further details of this involved but by now standard (in particular, nonsingular) argument. See also the comments of [WZ1, §1.5] on nonlocal reduction of the conservation law case to the standard one, which gives another route to this result via [WZ1, Thm. 1.2], which as noted in [WZ1] applies also to nonlocal equations. □

6.2. **Stability.** We next turn to the question of stability of exact periodic solutions of (1.2), and its relation with stability of periodic solutions (1.1) of (mcGL). Following [JNRZ], let $\mathcal{L}^{\varepsilon,\kappa,B_0}$ denote the linearized operator about an exact periodic traveling-wave solution $\bar u^{\varepsilon,\kappa,B_0}$ considered in a comoving frame for which it becomes stationary. The $L^2(\mathbb{R})$ spectrum of $\mathcal{L}^{\varepsilon,\kappa,B_0}$ consists of λ such that there exists a solution $v(x)$ on a single period of

$$\mathcal{L}^{\varepsilon,\kappa,B_0}v = \lambda v, \qquad x \in [0,X] = [0,1/\kappa], \tag{6.2}$$

satisfying for some Bloch-Floquet number $\sigma \in [-\pi/X, \pi/X)$ the boundary conditions

$$v(X)e^{i\sigma X} = v(0). \tag{6.3}$$

Diffusive spectral stability in the sense of Schneider is defined in this context as

$$\Re\lambda \le c(\varepsilon)|\sigma|^2/(1+|\sigma|^2) \quad \text{for } \lambda \in \operatorname{spec}(\mathcal{L}^{\varepsilon,\kappa,B_0}), \tag{6.4}$$

where $\nu = \varepsilon^2$. In the absence of neutral spectra other than the $m+1$ translational/conservational modes at $(\sigma,\lambda) = (0,0)$, (6.4) is necessary and sufficient for linearized and nonlinear stability [JNRZ]. By continuity of spectra for parabolic operators, diffusive spectral stability holds automatically for $0 < \varepsilon \le \varepsilon_0$ sufficiently small, except possibly for

$$|\lambda|, |\sigma| \le 1/C, \tag{6.5}$$

where $C > 0$ is arbitrary.

Expanding $\mathcal{L}^{\varepsilon,\kappa,B_0} = \mathcal{L}^{0,0,B_0} + O(\varepsilon)$, one may then perform as in [WZ3] a Lyapunov-Schmidt reduction, following the approach laid out in [M1, M2, M3, S1, S2], to obtain a reduced problem consisting in the rescaled $\check\lambda, \check\sigma$ variables of an $(m+2)\times(m+2)$ linear system of equations

$$\Big[-\check\lambda + \check M(\varepsilon,\check\sigma,\kappa,B_0) + \check{\mathcal{E}}(\varepsilon,\check\sigma,\kappa,B_0,\check\lambda)\Big]\begin{pmatrix} a_1\\ a_2\\ \vec b\end{pmatrix} = 0, \tag{6.6}$$

with $a_j \in \mathbb{R}$, $\vec b \in \mathbb{R}^m$, corresponding to the $m+2$ dimensional kernel of $\mathcal{L}^{0,0,B_0}$ at $\sigma = 0$, where $\check M$ is as in Section 4.

That is, first, the question of diffusive stability is reduced to determining diffusive spectral stability of the $m+2$ neutral eigenvalues branching as ε increases from zero from the kernel of the linearized operator about the constant state $(0, B_0)$ at bifurcation point $\varepsilon = 0$. And, second, these neutral eigenvalues may be identified as solutions of the reduced problem problem (6.6), a nonlinear eigenvalue problem that is a perturbation by $\mathcal{E}$ of the matrix perturbation problem corresponding to stability of traveling waves of (mcGL).

We collect here a streamlined version of the results established in [WZ3, Thm. 7.11 and Thm 8.7] and in the course of their proofs (in the case of Thm. 8.7, see the discussion just below).[3]

Proposition 6.2 (Truncation error bound [WZ3])**.** *For $0 < \varepsilon \le \varepsilon_0$ sufficiently small, diffusive stability of periodic solutions of $\bar u^{\varepsilon,\kappa,B_0}(kx+\bar\Omega t)$ is equivalent to diffusive stability for σ, λ arbitrarily small of $m+2$ "neutral" modes $\check\Lambda_j$ satisfying the reduced eigenvalue problem* (6.6)*, where $\check M$ as in Section 4 is the eigenvalue problem associated with* (mcGL)*, and*

$$|\check{\mathcal{E}}| = O(\varepsilon, \varepsilon^2\check\sigma, \varepsilon^3\check\sigma^2, \varepsilon^4\check\sigma^3, (\varepsilon^4+\varepsilon^6\check\sigma^2)\check\lambda^2). \tag{6.7}$$

[3]See also the explicit computations of [Wh, Appendix A.1] for an example model with $m = 1$.

Proof. In [WZ3], there is demonstrated the stronger truncation error bound

$$|\check{\mathcal{E}}| = O(\varepsilon^2, \varepsilon^2\check{\sigma}, \varepsilon^3\check{\sigma}^2, \varepsilon^4\check{\sigma}^3, (\varepsilon^4 + \varepsilon^6\check{\sigma}^2)\check{\lambda}^2)$$

for the "tilde model", a related, higher-order expansion of (mcGL) (denoted the "truncated model" in [WZ2]). Accounting for the difference in truncated vs. tilde models gives an additional $O(\varepsilon)$ error contribution, resulting in (6.7). See [Wh, Appendix A.1], for a detailed computation of errors for the truncated model in an example case with $m = 1$. □

Remark 6.3. Expressing (6.6) in the "original" coordinates $(\check{\lambda}, \check{\sigma}) = \varepsilon^{-2}(\lambda, \sigma)$ natural to PDE (1.2), as the system for (mcGL) perturbed by error $\mathcal{E}(\varepsilon, \sigma, \kappa, B_0, \lambda)$, (6.7) translates to

$$|\mathcal{E}| = O(\varepsilon^3, \varepsilon^2\sigma, \varepsilon\sigma^2, \sigma^3, (\varepsilon^2 + \sigma^2)\lambda^2). \tag{6.8}$$

The corresponding estimate for the tilde model becomes $|\mathcal{E}| = O(\varepsilon^4, \varepsilon^2\sigma, \varepsilon\sigma^2, \sigma^3, (\varepsilon^2 + \sigma^2)\lambda^2)$, and the difference between truncated and tilde models $O(\varepsilon^3)$.

The estimate (6.7) is effectively a "truncation error" or residual bound. What is needed to determine stability for (1.2) is to convert this truncation error to "convergence error," or difference between Λ_j and the solutions λ_j of the unperturbed system corresponding to stability for (mcGL), an issue requiring a detailed analysis of the eigenstructure of $\check{M}$. In this regard, it is worth mentioning that, while the estimates (6.7) come via (6.8) through Taylor expansion valid on the entire regime λ, σ small, the solutions $\check{\lambda}_j$ have Taylor expansions valid only only on the much smaller regime $\check{\lambda} = \sigma/\varepsilon^2$, $\check{\sigma} = \sigma/\varepsilon^2$ sufficiently small [WZ3, Thm. 5.9 and discussion in proof].

Such an analysis was carried out in [WZ3] on the region of analyticity $|\check{\sigma}| \le 1/C$ of Case (i), by essentially the same argument followed in Section 2. However, the corresponding treatment of remaining regions was left as an important and apparently difficult open problem. Indeed, the possibility of completing such an analysis was conjectured in [WZ3], somewhat optimistically, on the basis of numerical evidence, with a positive outcome far from clear, either for closeness of Λ_j and λ_j, or for determination of practical stability criteria for the λ_j.

The latter problem we have resolved in Section 4, showing that the stability requirements already determined in [WZ3] on the analyticity region $|\check{\sigma}| \le 1/C$ are in fact sufficient for stability on all regions. But, the $\check{\mathcal{E}} = 0$ case studied there was already a matrix perturbation problem, hence in the course of this analysis we have had to analyze the eigenstructure of the principal parts of $\check{M}$ in various regimes in order to absorb higher-order truncation errors arising from various spectral expansions. Hence, the analysis already completed for the second problem is also sufficient to resolve the first problem, provided only that we show that errors (6.7) are also absorbably small.

By this approach, we obtain as a corollary our second, and final, main theorem resolving stability.

Theorem 6.4 (Stability of exact solutions)**.** *Under the Turing hypotheses described above, with $\nu = \varepsilon^2$ and κ, B_0 satisfying* (1.8)*, let $\bar{u}^{\varepsilon,\kappa,B_0}(kx + \bar{\Omega}t)$ be the exact periodic solutions* (6.1) *of PDE* (1.2) *guaranteed by Theorem 6.1, and $(\bar{A}, \bar{B})^{\kappa,\omega} = (A_0 e^{i(\kappa x - \omega t)}, B_0)$ the corresponding periodic solutions* (1.1) *of the associated amplitude equations* (mcGL)*. Then, under the nondegeneracy conditions of Proposition 4.4, diffusive stability of $\bar{u}^{\varepsilon,\kappa,B_0}$ is equivalent to diffusive stability of $(\bar{A}, \bar{B})^{\kappa,B_0} = (A_0 e^{i(\kappa x - \omega t)}, B_0)$ for $0 < \varepsilon \le \varepsilon_0$ sufficiently small, which in turn is equivalent to the linear algebraic conditions* (1.20) *and* (1.23)-(1.25)*. Here, as elsewhere, ε_0 may be chosen uniformly for compact parameter sets on which the Turing and nondegeneracy assumptions are satisfied.*

Proof. Evidently, it suffices to establish necessity and sufficiency of conditions (1.20) and (1.23)-(1.25) for stability of $\bar{u}^{\varepsilon,\kappa,B_0}$, as we have shown already for stability of $(\bar{A}, \bar{B})^{\kappa,B_0}$. We study separately the regions described in cases (i)–(vi) of Section 4. To prove necessity of conditions (1.20) and (1.23)-(1.25), it is sufficient to show on just one of these regions, that eigenvalues Λ_j and λ_j are sufficiently close that their real parts have the same sign. To prove sufficiency, we must show this on each of the regions of cases (i)-(vi).

Case (i), ($|\sigma| \le \varepsilon^2/C$): In [WZ3, Thm. 8.7], it is shown for the tilde model that $|\Lambda_j - \lambda_j| = O(\varepsilon^4, \varepsilon\sigma, \varepsilon\sigma^2, \sigma^3)$ for all j; moreover [WZ3, Thm. 8.7, final line], for λ_t and λ_c, for which both $\lambda_j(\varepsilon, 0) = 0$ and $\Lambda_j(\varepsilon, 0) = 0$, $|\Lambda_j - \lambda_j| = O(\varepsilon\sigma, \varepsilon\sigma^2, \sigma^3)$. For, both λ_j and Λ_j are analytic on this regime, hence there can be no error term of order ε^4, nonvanishing at $\sigma = 0$. The same argument applies for the truncated model (mcGL) studied here, but with ε^4 order errors replaced by ε^3. Since both λ_j and Λ_j have first-order Taylor coefficient pure imaginary (also shown in [WZ3]), we have by similar reasoning that

$$|\Re\Lambda_j - \Re\lambda_j| = O(\varepsilon\sigma^2, \sigma^3) = o(\min\{\varepsilon^2, |\sigma|^2\}), \tag{6.9}$$

as there can be no error term in real parts of order $\varepsilon\sigma$ nonvanishing to order σ.

On the other hand, the analysis of Section 4, case (i) gives under our nondegeneracy assumptions that the stable eigenvalue λ_s has real part negative of order $-\varepsilon^2$, the translational eigenvalue λ_t has real part of size $\varepsilon^{-2}\sigma^2$, and the conservational eigenvalues λ_c have real part of size $\varepsilon^{-4}\sigma^2$ in the (original, pde) coordinates $(\lambda, \sigma) = \varepsilon^2(\check\lambda, \check\sigma)$; see (4.2)-(4.4). (Note, as $|\sigma| \ll \varepsilon^2$ on this region, that all $|\lambda_j| \ll 1$, despite large coefficients.) Comparing with (6.9), we thus see that the real parts of λ_j and Λ_j have common sign in each case.

The analysis of case (i), carried out previously in [WZ3], both establishes *necessity* of conditions (1.20) and (1.23)-(1.25), and reduces the study of *sufficiency* to the examination of cases (ii)-(vi). In the rest of the proof, we carry out this remaining part, showing sufficiency in each of the cases (ii)-(vi).

Case (ii). ($1/C \le |\check\sigma| \le C$): This case for (mcGL) is about continuity of spectra of M_0 with respect to "errors" ρM_1 and $\rho^2 M_2$ that are merely $o(1)$. So, to accomodate also the additional error $\check{\mathcal{E}}$, we have only to show that (6.7) is small on this regime, and absorbably in the errors induced by M_1 and M_2. We first observe, by $\check\lambda V = o(|\check\lambda|)V + (M_0 + o(1))V$ for $V \ne 0$ that $(1 + o(1))|\check\lambda| \le |M_0 + o(1)| \lesssim C$, hence $\check\lambda = O(1)$, and thus by (6.7)

$$|\check{\mathcal{E}}| = O(\varepsilon, \varepsilon^2\check\sigma, \varepsilon^3\check\sigma^2, \varepsilon^4\check\sigma^3, (\varepsilon^4 + \varepsilon^6\check\sigma^2)\check\lambda^2) = O(\varepsilon^2, \varepsilon^2\check\sigma, \varepsilon^3\check\sigma^2, \varepsilon^4\check\sigma^3, o(\check\lambda)) = o(1). \tag{6.10}$$

Thus, the induced errors are indeed small, preserving the order one separation between the kernel of M_0 and remaining eigenvalues featuring a spectral gap. It follows that the "small" eigenvalue bifurcating from the kernel remains analytic in ρ, uniformly in $\check\sigma$, and the remaining "order one" eigenvalues possess a spectral gap, having real parts negative and uniformly bounded from zero.

By the $o(1)$ error estimate (6.10), the order one eigenvalues with spectral gap retain this spectral gap under perturbation, by simple continuity of spectra, and so $\Re\Lambda_j$ and $\Re\lambda_j$ have the same signs for these modes, carrying the same stability information.

For the remaining "small" eigenvalue λ_t, we have $\Re\lambda_t \sim \rho^2 = \varepsilon^2\check\sigma^2$, we must look more carefully, as potential errors of order ε^2 or $\varepsilon^2\check\sigma$ are much greater than $\Re\lambda_t$ near the lower boundary $|\check\sigma| = 1/C$ where $|\check\sigma| \ll 1$. However, the same reasoning shows that such error terms therefore cannot occur, by matching at the boundary with region (i). For, as we have demonstrated analyticity of the projector onto the small eigenmode, and as the initial error $\check{\mathcal{E}}$ is analytic on all $|\sigma| \ll 1$, we have that the projected error $\check{\mathcal{E}}_t$ in the reduced problem for Λ_t is analytic as well, as a function of $\check\sigma, \varepsilon, \rho$, for σ in region (ii), and indeed for complexified $\check\sigma$, ε, for $1/C \le |\sigma| \le C$ and $|\varepsilon| \ll 1$. But, by the balancing transformation argument in region (i), we have already shown that the projector onto the small, λ_t mode is analytic in (complexified) σ, ε for for $|\sigma| \le 1/C$ and $|\varepsilon| \ll 1$, hence the projector, and projected error are analytic for $|\sigma| \le C$ and $|\varepsilon| \ll 1$, so that *λ_t and Λ_t are analytic as well, with convergent power series representations about* $(\sigma, \varepsilon) = (0, 0)$. We may thus obtain the desired sharpened estimates from the power series analysis of region (i), which shows that "harmful" order ε^4 or $\varepsilon^2\sigma$ terms do not occur.

Note, in showing analyticity of the complexified λ_t projector on region (ii), we require that $\hat M_0$ in (4.5) have a kernel of dimension one, i.e., that the complementary matrix

$$\hat M_0 := LM_0R = \begin{pmatrix} 2A_0^2\Re(c) & A_0\Re(d) \\ 2A_0 h\hat{i\sigma} & f\hat{i\sigma} \end{pmatrix}$$

derived in (5.23) be invertible for $\sigma \neq 0$. But, direct computation gives

$$\det \hat M_0 = 2A_0^2 i\sigma \Re(c) \det\Big(f - h\Re(d)\Re(c)\Big) \neq 0$$

for $\sigma \neq 0$, since $\Re(c) = 0$ and $\det(f - h\Re(d)\Re(c)) = 0$ are excluded by our previous nondegeneracy assumptions.

Case (iii). ($C \leq |\check\sigma| \leq 1/C\varepsilon$): Similarly as in Case (ii), the treatment of Case (iii) in Section 4 requires only that errors be $o(1)$, much smaller than spectral separation between small eigenvalue λ_t and remaining eigenvalues. Starting with $\check\lambda V = o(|\check\lambda|)V + (M_0 + o(1))V$ for $V \neq 0$, giving $(1 + o(1))|\check\lambda| \leq |M_0 + o(1)| \lesssim \check\sigma$, hence $\check\lambda = O(\check\sigma)$, we find that

$$\varepsilon^4|\check\lambda|^2 = O(\varepsilon^4\check\sigma^2) = O(1/C) = o(1).$$

Likewise,

$$O(\varepsilon, \varepsilon^2\check\sigma, \varepsilon^3\check\sigma^2, \varepsilon^4\check\sigma^3, \varepsilon^4\check\sigma^2)) = O(\varepsilon, \varepsilon/C, \varepsilon/C^2, \varepsilon/C^3, \varepsilon^2/C^2) = o(1),$$

confirming smallness of remaining error terms. Finally, we note that the "small" λ_t mode has real part $\sim \rho^2 = \varepsilon^2|\check\sigma|^2 \gg \varepsilon^2|\check\sigma|$ on this region, hence can absorb all error terms other than constant, $O(\varepsilon)$ ones. Observing that analyticity of λ_t and Λ_t holds up to $|\rho| \ll 1$, by the same argument as in region (ii), we find again that such an error term cannot exist.

Case (iv). ($1/C\varepsilon \leq |\check\sigma| \leq 1/C\varepsilon^2$): This case is straightforward. For, both symbol and real parts of eigenvalues are of the same order $(1 + |\rho|^2)$, hence we need only show that truncation errors are order $o(1 + |\rho|^2)$ in order to see by simple continuity of eigenvalues under perturbation that $\Re\Lambda_j$ and $\Re\lambda_j$ have the same signs.

Here, $|\rho| = \varepsilon|\check\sigma| \geq 1/C$, while $|\check\sigma| \gg 1$ and $\varepsilon|\rho| = \varepsilon^2\check\sigma| \ll 1$. Thus, λ-errors in $\check{\mathcal{E}}$ are bounded by $\varepsilon^4|\check\lambda|^2 = |\lambda|^2 = o(1)$ and $\varepsilon^6|\check\sigma|^2|\check\lambda|^2 = |\rho|^2\varepsilon^4|\check\lambda|^2 = o(\rho^2)$, both $o(1 + |\rho|^2)$. Likewise, the remaining errors in $\check{\mathcal{E}}$ are bounded by

$$O(\varepsilon, \varepsilon^2\check\sigma, \varepsilon^3\check\sigma^2, \varepsilon^4\check\sigma^3, \varepsilon^4\check\sigma^2)) = o(1, 1, |\rho|^2, |\rho|^2, |\rho|^2) = o(1 + |\rho|^2),$$

so again are absorbable.

Case (v). $1/C\varepsilon^2 \leq |\check\sigma| \leq C/\varepsilon^2$: This case was carried out for (mcGL) in the "original PDE coordinates" λ, σ coordinates with $1/C \leq |\sigma| \leq C$, featuring a spectral gap of order $\sigma^2 \geq 1/C^2$ for eigenvalues λ_j. The error (6.8) of $O(\varepsilon^3, \varepsilon^2\sigma, \varepsilon\sigma^2, \sigma^3, \lambda^2)$ need thus be only $o(\sigma^2)$, or, sufficiently, $o(1)$. Recalling that $|\lambda| = o(1)$, we see that this is evidently so.

Case (vi). $C/\varepsilon^2 \leq |\check\sigma|$: In this case, working with the original PDE coordinates $(\sigma, \lambda) = \varepsilon^2(\check\sigma, \check\lambda)$, we have $|\sigma| \geq C$. Recall that this case is always stable for (mcGL). Likwise, as noted in (6.5), the spectra for the exact PDE problem is also automatically stable in this regime. Thus, stability of λ_j and Λ_j are (trivially) equivalent. □

Remark 6.5. An examination of the analyticity argument for the projector onto the λ_t mode in region (ii) shows that the argument can in fact be extended also to all of region (iii). This shows that λ_t and Λ_t have convergent analytic expansions in $\hat\sigma$, ε for $|\hat\sigma| \leq 1/C$ and $\varepsilon \ll 1$, similarly as in the nonconservative case. In particular, it gives an additional verification that the second order coefficient μ_t^0 derived on region (i) is valid also on region (iii), for which the Dary approximation is valid. This gives an additional proof that the descriptions of λ_t behavior for Eckhaus and Darcy approximations agree.

Appendix A. Turing bifurcation for example model

For the model problem (1.4). homogeneous equilibrium states take the form (ρ_0, u_0, c_0), with

$$u_0 = 0, \quad c_0 = \alpha\tau\rho_0, \tag{A.1}$$

without loss of generality (rescaling parameters if necessary) $\rho_0 = 1$. For simplicity in writing, let us fix $\mu = \nu = 0$; as we note at the end, this does not affect the end result.

Linearized about such a state, (1.4) then becomes

$$\begin{pmatrix}\rho\\ u\\ c\end{pmatrix}_t = \begin{pmatrix}0 & 0 & 0\\ 0 & -\gamma & 0\\ \alpha & 0 & -1/\tau\end{pmatrix}\begin{pmatrix}\rho\\ u\\ c\end{pmatrix} + \begin{pmatrix}0 & -1 & 0\\ -2A & 0 & \beta\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\rho\\ u\\ c\end{pmatrix}_x + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & D\end{pmatrix}\begin{pmatrix}\rho\\ u\\ c\end{pmatrix}_{xx}. \tag{A.2}$$

The associated dispersion relation $\lambda = \lambda(k)$ is thus given by

$$0 = \det\begin{pmatrix}-\lambda & -ik & 0\\ -2Aik & -\gamma-\lambda & \beta ik\\ \alpha & 0 & -k^2D - 1/\tau - \lambda\end{pmatrix}. \tag{A.3}$$

Examining (A.3) at $k = 0$, we see that the eigenvalues are $0, -\gamma, -1/\tau$, so that there is a single "critical" analytic branch $\lambda_*(k)$ with $\lambda_*(0) = 0$. Moreover, it is readily seen by matching common orders of k that λ_* vanishes to first order as well. Substituting

$$\lambda_*(k) = \theta k^2 + O(k^3)$$

into (A.3) and collecting terms of order k^2, we find that $(-1/\tau)(\theta k^2\gamma + Ak^2) + \beta\alpha k^2 = 0$, or

$$\theta = \alpha\beta\tau - A. \tag{A.4}$$

Thus, diffusive stability of the dispersion relation near $k = 0$ is equivalent to

$$A > \alpha\beta\tau. \tag{A.5}$$

On the other hand, for $\alpha\beta = 0$, the linearized matrix becomes block triangular, and we can solve the dispersion relation explicitly as

$$\lambda = -1/\tau - Dk^2, \frac{-\gamma \pm \sqrt{\gamma^2 - 8Ak^2}}{2}, \tag{A.6}$$

yielding strict diffusive stability by inspection. Thus, strict diffusive stability holds for $\alpha\beta$ sufficiently small, while, by (A.5), fails for $\alpha\beta$ sufficiently large.

We may conclude therefore that a Turing bifurcation occurs for some value $(\alpha\beta)_* > 0$. Essentially the same argument yields the result for arbitrary $\mu, \nu > 0$ as well. It would be very interesting to determine the nature and properties of this bifurcation, in the spirit of the current analysis.

Appendix B. Numerical illustration, case $m = 1$

In the below figures, we display the results of a numerical comparison of the spectra of the linearized equations for (mcGL) and the results predicted by Taylor expansion, for a generic example system with $m = 1$. The parameters chosen are $a = 1 + i$, $b = 1$, $c = -3 + 2i$, $d = -1 + 2i$, $e_B = 1$, $f = 1$, $g = 2 + 2i$, $h = 2$, $\varepsilon = 10^-2$. In the illustrations, blue dots denote the real parts of (numerically approximated) true eigenvalues, green, predictions from μ_t, and red, predictions from μ_c, with lefthand panel displaying results for small σ and righthand large σ, and each pair of panels associated with a different wave-number κ.

In the left column, we've plotted the curves on $|\hat\sigma| \leq 10\varepsilon$ and in the right column, we have the curves on $|\hat\sigma| \leq 1$. We note the strong agreement between $\mu_c\hat\sigma^2$ and $\Re\lambda_c(\hat\sigma)$ on $\hat\sigma = o(\varepsilon)$ and good agreement between $\mu_t\hat\sigma^2$ and $\Re\lambda_t(\hat\sigma)$ on the region where $\hat\sigma = o(1)$, as is expected by

Remark 6.5. We emphasize that the singularity in the dispersion relations are quite prominent, as $\Re\lambda_c(\hat{\sigma})$ reaches $O(1)$ when $\hat{\sigma} \sim \varepsilon$, in contrast to the classical Ginzburg-Landau or Matthews-Cox cases where $\Re\lambda_c(\hat{\sigma}) \sim \varepsilon^2$ when $\hat{\sigma} \sim \varepsilon$. Note also that the effects of singularity are visible in the right-hand column of figures through the "spikes" near $\hat{\sigma} = 0$. The final point to which we wish to draw emphasis is that it is λ_c and λ_s that intersect and lose analyticity within an $O(\varepsilon)$ domain, while λ_t remains analytic on a domain of $o(1)$. We recall that this played a key role in the analysis of Cases (ii)-(iv). One final remark is that, for fixed ε, the agreement between $\mu_t\hat{\sigma}^2$ and $\lambda_t(\hat{\sigma})$ gets worse as $|\kappa|$ increases. This is to be expected, as the formula for μ_t in (2.52) goes to ∞ as $|\kappa|^2 \to \kappa_E^2$.

In the figures below, we have chosen the frequencies $\kappa = 0$, $\kappa = \kappa_E/4(= 0.25)$ and $\kappa = \kappa_E/2 = (0.5)$ as the numerically observed stability boundary occurs at around $\kappa \approx \pm 0.3$.[4]

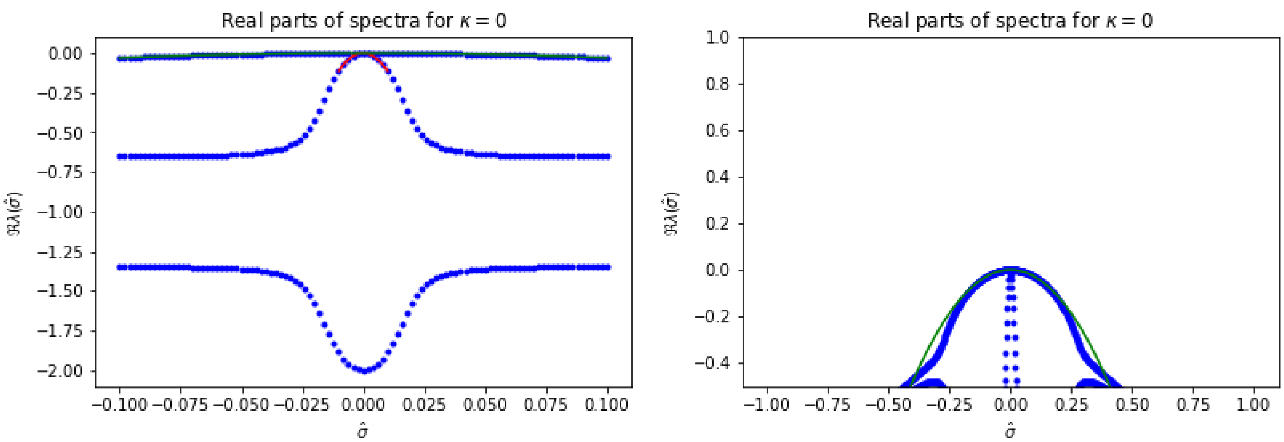

FIGURE 1. Wavenumber $\kappa = 0$.

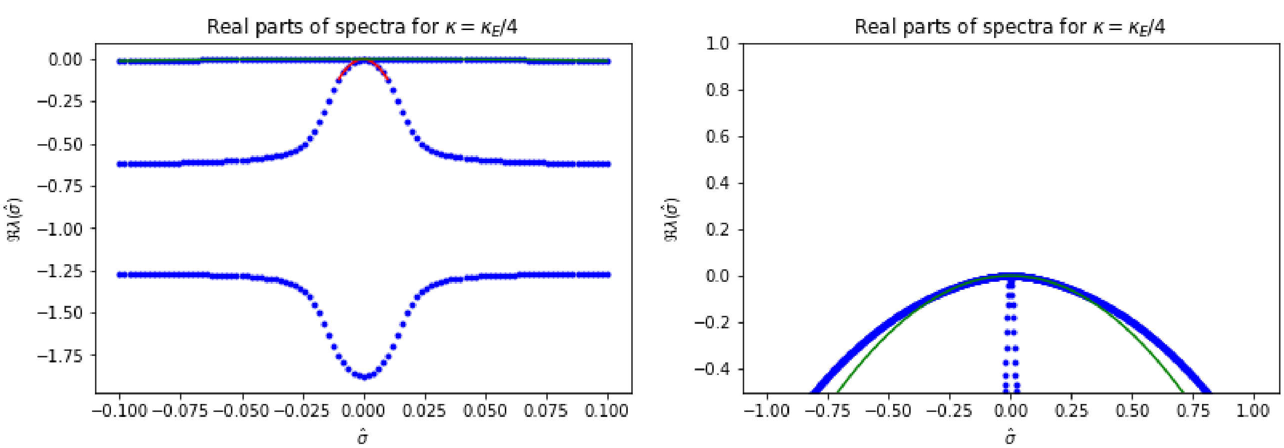

FIGURE 2. Wavenumber $\kappa = \kappa_E/4$.

[4]Close to the real Ginzburg-Landau boundary of 1/3.

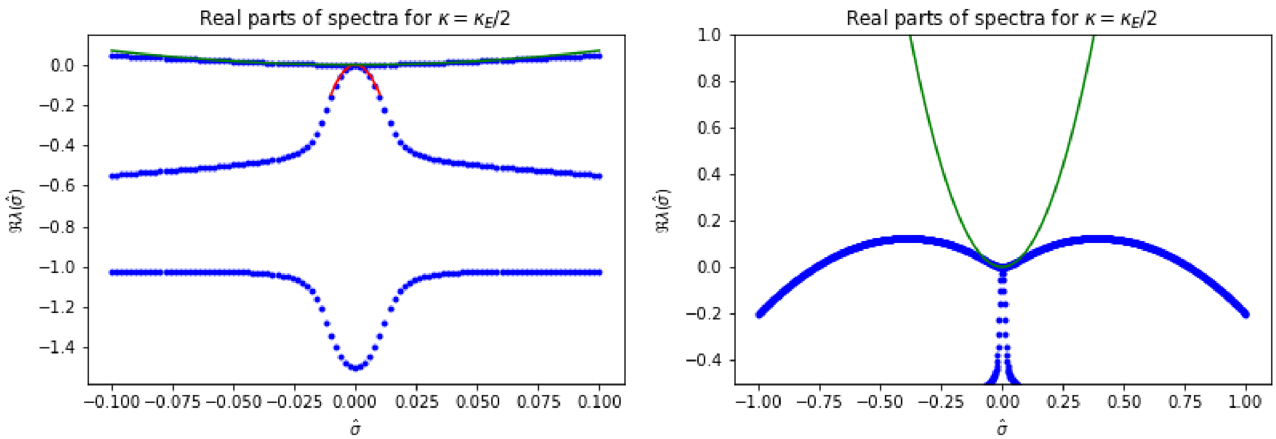

FIGURE 3. Wave number $\kappa = \kappa_E/2$.

APPENDIX C. MATCHED ASYMPTOTICS FOR COMPLEX GINZBURG-LANDAU

Here, we compute the expansion for the neutral translational mode for complex Ginzburg-Landau using matched asymptotics. To do so, we let a, b, c be known complex numbers with $\Re(a), \Re(b) > 0$ and $\Re(c) < 0$ and consider a generic Ginzburg-Landau equation of the form

$$A_T = aA_{XX} + bA + c|A|^2 A.$$

As before, we have a one-parameter family of periodic traveling waves parameterized by κ of the form $A(X,T) = A_0 e^{i(\kappa X - \omega T)}$ with $A_0 > 0$. The same linearization procedure as in [WZ2] leads us to the spectral problem

$$\lambda \begin{pmatrix} u \\ v \end{pmatrix} = -\sigma^2 \begin{pmatrix} \Re(a) & -\Im(a) \\ \Im(a) & \Re(a) \end{pmatrix} + 2i\kappa\sigma \begin{pmatrix} -\Im(a) & -\Re(a) \\ \Re(a) & -\Im(a) \end{pmatrix} + \begin{pmatrix} 2A_0^2\Re(c) & 0 \\ 2A_0^2\Im(c) & 0 \end{pmatrix} =: m(\sigma) \begin{pmatrix} u \\ v \end{pmatrix}.$$

We then seek an eigenvalue λ_t of the form $\lambda_t(\sigma) = i\alpha_t\sigma + \mu_t\sigma^2 + O(\sigma^3)$. We notice that the second column of $m(\sigma) - \lambda_t(\sigma)Id$ is proportional to $i\sigma$, and so our expansion of the determinant is of the form

$$\det(m(\sigma) - \lambda_t(\sigma)Id_2) = i\sigma\Big(\det P_0 + i\sigma \det P_1 + H.O.T.\Big),$$

with

$$P_0 = \begin{pmatrix} 2A_0^2\Re(c) & -2\kappa\Re(a) \\ 2A_0^2\Im(c) & -2\kappa\Im(a) - \alpha_t \end{pmatrix},$$

and

$$P_1 = \begin{pmatrix} -2\kappa\Im(a) - \alpha_t & -2\kappa\Re(a) \\ 2\kappa\Re(a) & -2\kappa\Im(a) - \alpha_t \end{pmatrix} + \begin{pmatrix} 2A_0^2\Re(c) & -\Im(a) \\ 2A_0^2\Im(c) & \Re(a) + \mu_t \end{pmatrix}.$$

Setting $\det P_0 = \det P_1 = 0$ and solving the corresponding equations gives

$$\alpha_t = -2\kappa\Im(a) + 2\kappa\Re(a)\frac{\Im(c)}{\Re(c)},$$

and

$$\mu_t = -\frac{\big(-2\kappa\Im(a) - \alpha_t\big)^2 + 4\kappa^2\Re(a)^2 + 2A_0^2\big(\Re(a)\Re(c) + \Im(a)\Im(c)\big)}{2A_0^2\Re(c)}.$$

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